Long-term Causal Inference Under Persistent Confounding via Data Combination

Guido Imbens¹*, Nathan Kallus², Xiaojie Mao³, Yuhao Wang^{3,4}

¹Stanford University; ²Cornell University; ³Tsinghua University; ⁴Shanghai Qi Zhi Institute.

Abstract

We study the identification and estimation of long-term treatment effects by combining short-term experimental data and long-term observational data subject to unobserved confounding. This problem arises often when concerned with long-term treatment effects since experiments are often short-term due to operational necessity while observational data can be more easily collected over longer time frames but may be subject to confounding. In this paper, we uniquely tackle the challenge of persistent confounding: unobserved confounders that can simultaneously affect the treatment, short-term outcomes, and long-term outcome. In particular, persistent confounding invalidates identification strategies in previous approaches to this problem. To address this challenge, we exploit the sequential structure of multiple short-term outcomes and develop three novel identification strategies for the average long-term treatment effect. Based on these, we develop estimation and inference methods with asymptotic guarantees. To demonstrate the importance of handling persistent confounders, we apply our methods to estimate the effect of a job training program on long-term employment using semi-synthetic data.

Keywords: data combination, doubly robust estimation, long-term causal inference, proxy variables, unobserved confounding.

1 Introduction

Empirical researchers and decision-makers are often interested in learning the long-term treatment effects of interventions. For example, labor economists are interested in the effect of early childhood education on lifetime earnings [Chetty et al., 2011], marketers are interested in the effects of promotions on long-term revenue [Yang et al., 2020a], online platforms are interested in the effects of webpage designs on users' long-term behaviors [Hohnhold et al., 2015]. Since a long-term effect can be quite different from short-term effects [Kohavi et al., 2012], accurately evaluating the long-term effect is both difficult and crucial for comprehensively understanding the intervention of interest.

Learning long-term treatment effects is very challenging in practice because long-term outcomes are seldom observed within the time frame of randomized experiments. For example, randomized experiments in online platforms (often termed A/B tests within that context) usually last for only a few weeks, and practitioners in the industry commonly recognize evaluation of long-term effects as a paramount challenge [Gupta et al., 2019]. In contrast, observational data are often easier and cheaper to acquire and can be collected retroactively, so they are more likely to include long-term

^{*}Corresponding author: imbens@stanford.edu

[†]Alphabetical order.

outcome observations. Nevertheless, observational data are susceptible to unmeasured confounding, which can lead to biased treatment effect estimates. Therefore, long-term causal inference is very challenging using only experimental or observational data, either due to missing long-term outcome (in experimental data) or unmeasured confounding (in observational data).

In this paper, we study the identification and estimation of long-term treatment effects by combining both experimental and observational data. By combining these two different types of data, we hope to leverage their complementary strengths, i.e., the randomized treatment assignments in the experimental data and the long-term observations in the observational data. In particular, we aim to tackle the presence of persistent confounding in the observational data, which cannot be generally ruled out. That is, we allow some unobserved confounders to have persistent effects in the sense that they can affect not only the short-term outcomes but also the long-term outcome. Persistent confounders are prevalent in long-term studies. For example, in studying early childhood education's effect on lifetime earnings, students' innate intelligence and/or familial support systems can affect both short-term and long-term earnings. Our setup is summarized in the causal diagrams in Figure 1.

A few previous works also consider data combination for long-term causal inference. Athey et al. [2019], in a setting where the observational sample contains no information on the treatment, rely on a surrogate criterion first proposed by Prentice [1989]. Athey et al. [2020], in the same setting as considered in the current paper, assume a latent unconfoundedness condition. While these conditions make no explicit reference to persistent confounding and its absence, a nontrivial persistent confounder can generally violate these (see Appendix A for details). At the same time, both settings are *just identified*, meaning the conditions imposed are minimal. Therefore, some other conditions are needed in their place to guarantee identification while permitting more general persistent confounding.

In this paper, we leverage an assumed sequential structure between multiple short-term outcomes to tackle long-term causal inference in the presence of persistent confounders. Our new identification and estimation strategies are based on using short-term outcomes as proxy variables for the persistent confounders. To the best of our knowledge, this is the first time that the internal structure of short-term outcomes is used to address unmeasured confounding in long-term causal inference. Indeed, although Athey et al. [2019, 2020] also advocate using multiple short-term outcomes, they view them as a whole without leveraging their internal structure. Our work therefore also provides new insights on the special role of using multiple short-term outcomes in long-term causal inference.

Our contributions are summarized as follows:

- We propose three novel identification strategies for the average long-term treatment effect in the presence of persistent confounders. These identification strategies rely on three groups of short-term outcomes, where two of these groups are used as informative proxy variables for the unobserved confounders (Assumption 5). These short-term outcomes, together with the long-term outcome, are assumed to follow a sequential structure encapsulated in a conditional independence condition (Assumption 4).
- Based on each of the three identification strategies, we propose corresponding estimators
 for the average long-term treatment effect. These estimators involve fitting two nuisance
 functions that are defined as solutions to two conditional moment equations. Our estimation
 procedures accommodate any nuisance estimator among many existing ones. We provide high
 level conditions for the asymptotic consistency and asymptotic normality of our estimators.
- We evaluate the performance of our proposed estimators based on large-scale experimental

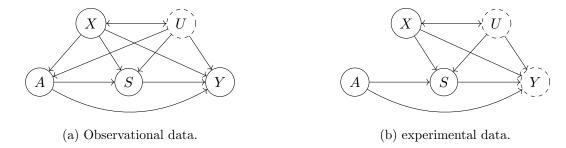


Figure 1: Causal diagrams for observational and experimental data with persistent confounders. Here A denotes the treatment, S denotes (multiple) short-term outcomes, Y denotes the long-term outcome, X denotes covariates, and U denotes unobserved confounders. Confounders U in both samples and the long-term outcome Y in the experimental data are unobserved, so they are indicated by dashed circles. Note that unobserved confounders U can simultaneously affect short-term outcomes S and the long-term outcome Y.

data for a job-training program with long-term employment observations. We combine part of the experimental data and some semi-synthetic observational data with realistic persistent confounding. We demonstrate that due to the persistent confounding, our proposed estimators have smaller error than estimators that do not handle persistent confounding.

The rest of this paper is organized as follows. We first review the related literature in Section 2 and set up our problem in Section 3. Then we discuss our identification strategies in Section 4, where each subsection features one different identification strategy. In Section 5, we present our long-term treatment effect estimators and analyze their asymptotic properties. We further dicuss some extensions in Section 6. In Section 7, we illustrate the performance of methods in a semi-synthetic experiment. We finally conclude this paper in Section 8.

2 Related Literature

2.1 Surrogates

Our paper is related to a large body of biostatistics literature on surrogate outcomes; see reviews in Weir and Walley [2006], VanderWeele [2013], Joffe and Greene [2009].

These literature consider using the causal effect of an intervention on a surrogate outcome (e.g., patients' short-term health) as a proxy for its treatment effect on the outcome of primary interest (e.g., long-term health). To this end, many criteria have been proposed to ensure the validity of the surrogate outcome. Examples include the statistical surrogate criterion [Prentice, 1989], principal surrogate criterion [Frangakis and Rubin, 2002], consistent surrogate criterion [Chen et al., 2007], among many others. However, these criteria can easily run into a logical paradox [Chen et al., 2007] or rely on unidentifiable quantities, showing the challenge of causal inference when the primary outcome is completely missing. When multiple surrogates are available, Wang et al. [2020], Price et al. [2018] consider transforming these surrogates to optimally approximate the primary outcome. Their approaches can avoid the surrogate paradox discussed in Chen et al. [2007]. Nevertheless, learning surrogate transformations requires having experimental data with long-term outcome observations.

In contrast, our paper does not need long-term outcome observations in the experimental data but only need them in observational data. Moreover, our paper does not view short-term outcomes as proxies for the long-term outcome, so we avoid these previous surrogate criteria. Instead, we view them as proxies for unobserved confounders to correct for confounding bias. See also discussions in Section 2.3.

2.2 Data Combination for Long-term Causal Inference and Decision-Making

Following Athey et al. [2019], some recent literature also combine experimental and observational data, and rely on the statistical surrogate criterion, either to estimate cumulative treatment effects in dynamic settings [Battocchi et al., 2021] or learn long-term optimal treatment policies [Yang et al., 2020a, Cai et al., 2021b]. Chen and Ritzwoller [2021] derive the efficiency lower bound for average long-term treatment effect in settings of Athey et al. [2019] and Athey et al. [2020]. Singh [2021, 2022] further develop debiased long-term treatment effect estimators based on machine learning nuisance estimation. In contrast, Kallus and Mao [2020], Cai et al. [2021a] combine two datasets that both satisfy unconfoundedness. Still, all of these works rule out persistent confounding, which is the main problem tackled in this paper.

A concurrent and independent work by Ghassami et al. [2022] uses alternative conditions or additional variables to alleviate latent confounding in long-term causal inference. They propose three different identification strategies, and their proximal data fusion strategy is closely related to our approach in Sections 4.1 and 4.2 and appendix D.1. Their approach requires auxiliary proxy variables satisfying certain generic conditions (in addition to the short-term outcomes). In contrast, our work specifically leverages the special sequential structure of multiple short-term outcomes and shows how such short-term outcomes can proxy the confounders. This obviates the need to search for external proxy variables and allows us to understand the different types of confounders and which need to be controlled (see appendix E.2). Importantly, we develop both estimation and inference methods with theoretical guarantees and validate them in a concret case study. Moreover, we provide an alternative control function identification strategy in Section 6.2 and study how the short-term outcomes may help weaken a widely assumed external validity condition in Appendix E.3. These results have no analogues in Ghassami et al. [2022].

There is also growing interest in combining experimental and observational data to improve, rather than enable, causal inference [e.g., Chen et al., 2021, Cheng and Cai, 2021, Yang et al., 2020b,c, Colnet et al., 2020, Kallus et al., 2018, Rosenman et al., 2022, 2020, Yang and Ding, 2019]. In these works, the outcome of interest is observed in both types of data, so causal-effect identification is already guaranteed by the experimental data. Instead, the aim of the data combination is to reduce variance. In contrast to these works, in our setting, data combination is crucial for causal identification since any one data set alone cannot identify the long-term treatment effect.

2.3 Proximal Causal Inference

Our identification proposals are related to how proximal causal inference deals with unmeasured confounding by leveraging proxy variables [Tchetgen Tchetgen et al., 2020]. The seminal work of Miao et al. [2016] demonstrated the identification of treatment effects with unobserved confounders given two different types of proxy variables: negative control outcomes, which are not affected by the treatment, and negative control treatments, which do not affect the outcome. Since then, a series of works have proposed a variety of different estimation methods based on this identification strategy [Kallus et al., 2021, Ghassami et al., 2021b, Deaner, 2021, Singh, 2020, Miao and Tchetgen, 2018, Shi et al., 2020, Mastouri et al., 2021, Cui et al., 2020]. The proximal causal inference framework has also been extended to longitudinal data analysis [Imbens et al., 2021, Ying et al., 2021, Shi et al., 2021], mediation analysis [Dukes et al., 2021, Ghassami et al., 2021a], and off-policy evaluation and learning [Bennett and Kallus, 2021, Tennenholtz et al., 2020, Qi et al., 2021, Xu et al., 2021].

The existing proximal causal inference literature focus on a single observational dataset. In contrast, in this paper we consider combining observational and experimental data. We view short-term outcomes as proxy variables for persistent unmeasured confounders. However, all of these short-term outcomes can be affected by the treatment (see Figure 3 below), so they do not satisfy the proxy conditions in Miao et al. [2016]. In this paper, we establish novel identification strategies that leverage the additional experimental data. See also discussions in Remark 2.

3 Problem Setup

We consider a binary treatment variable $A \in \mathcal{A} = \{0,1\}$ where A = 1 stands for the treated group and A = 0 stands for the control group. We are interested in the treatment effect on a long-term outcome. Based on the potential outcome framework [Rubin, 1974], we postulate potential long-term outcomes $Y(0) \in \mathcal{Y} \subseteq \mathbb{R}$ and $Y(1) \in \mathcal{Y} \subseteq \mathbb{R}$, which would be realized were the treatment assignment equal 0 and 1, respectively. In reality, we can observe at most one of the potential outcomes per unit, corresponding to the actual treatment assignment, Y = Y(A).

We may in fact observe neither potential long-term outcome in short-term experiments that end before these long-term outcomes can be observed. Nevertheless, it is usually still possible to observe some short-term outcomes. We postulate potential short-term outcomes $S(1) \in \mathcal{S}, S(0) \in \mathcal{S}$, and denote the observable realized short-term outcomes as S = S(A). In this paper, we consider multiple short-term outcomes, so we generally understand S as a vector. We discuss our assumptions on the inner structure of these short-term outcomes in section 3.2. Additionally, we can observe some pre-treatment covariates denoted as $X \in \mathcal{X}$.

We have access to two samples: an observational (O) sample with n_O units and an experimental (E) sample with n_E units. We suppose that the observational sample is a random sample from the population of interest, where for each unit i we can observe independently and identically distributed tuples (X_i, A_i, S_i, Y_i) . The experimental sample may be a selective sample from the same population, where for each unit i we only observe (X_i, A_i, S_i) , but not the long-term outcome. We use a binary indicator $G_i \in \{E, O\}$ to denote which sample a unit i belongs to. Without loss of generality, we can consider a combined i.i.d sample of size $n = n_O + n_E$ from an artificial superpopulation, namely, $\mathcal{D} = \{(G_i, X_i, A_i, S_i, Y_i \mathbb{I}[G_i = O]) : i = 1, \ldots, n_O + n_E\}$. We use \mathbb{P} and \mathbb{E} to denote the probability measure and expectation with respect to this super-population, and use $p(\cdot)$ to denote the associated probability density function or probability mass function, as appropriate. We also denote the observational and experimental subsamples as \mathcal{D}_O and \mathcal{D}_E , respectively.

Our aim is to combine the observational and experimental samples in order to learn the long-term treatment effect on the population associated with the observational data:

$$\tau = \mu(1) - \mu(0),$$
 where $\mu(a) = \mathbb{E}[Y(a) \mid G = O].$ (1)

Our results easily extend to the average on the experimental or combined population. We focus on τ above for concreteness and we believe it captures the most commonly relevant estimand.

3.1 Basic Assumptions Characterizing the Observational and Experimental Data

We now describe the basic assumptions that characterize the experimental and observational data sets as such. Unless otherwise stated, all of these assumptions are maintained throughout this paper.

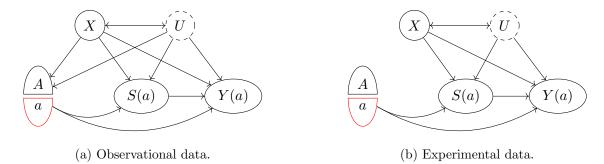


Figure 2: Single world intervention graphs (SWIG) corresponding to the causal diagrams in Figure 1.

The observational data is generally confounded, that is, conditioning only on X does not render the treatment assignment independent of the potential short-term and long-term outcomes. Instead, there exist some unobserved confounders $U \in \mathcal{U}$ that are needed to account for the association between treatment and potential outcomes. See Figure 2a for a single world intervention graph illustration [Richardson and Robins, 2013] when intervening on the variable A.

Assumption 1 (Observational data). For $a \in \{0, 1\}$,

$$(Y(a), S(a)) \perp A \mid U, X, G = O, \tag{2}$$

and $0 < \mathbb{P}(A = 1 \mid U, X, G = O) < 1$ almost surely.

Equation (2) means that U and X together account for all confounding in the observational data, and generally the observed covariates X alone are not enough. Moreover, we impose the overlap condition $0 < \mathbb{P}(A=1 \mid U, X, G=O) < 1$, which is a standard assumption in causal inference literature. Note that the existence of U is without loss of generality because we can always take it to be the potential outcomes themselves. Because of the unobserved confounders U, the observational data alone is not enough to identify the treatment effect parameter τ in eq. (1).

In contrast to the observational data, the treatment assignments are assigned completely at random in the experimental data. See Figure 2b for a single world intervention graph illustration.

Assumption 2 (Experimental Data). For $a \in \{0, 1\}$,

$$(Y(a), S(a), U, X) \perp A \mid G = E, \tag{3}$$

and $0 < \mathbb{P}(A = 1 \mid G = E) < 1$ almost surely.

Although unconfounded, the experimental data do not contain long-term outcome observations, so the experimental data alone is not enough to identify the treatment effect either. This motivates us to combine the observational and experimental data. To this end, we further impose the following assumption permitting such combination.

Assumption 3 (External Validity). For any $a \in \{0, 1\}$,

$$(S(a), U, X) \perp G, \tag{4}$$

and, almost surely,

$$\frac{p(U,X\mid A=a,G=E)}{p(U,X\mid A=a,G=O)}<\infty. \tag{5}$$

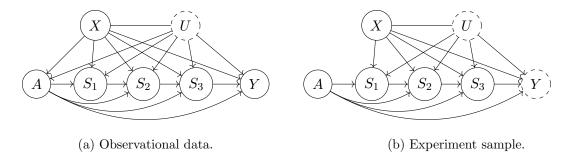


Figure 3: Sequential structure of three groups of short-term outcomes.

Assumption 3 ensures that the two samples have enough commonality so it is meaningful to combine them. Equation (4) in Assumption 3 means that the experimental data has external validity, in that the distribution of (S(a), U, X) in the experimental data is the same as that in the observational data (i.e., the population of interest). Similar assumptions also appear in previous literature that attempt to combine different samples [e.g., Athey et al., 2020, 2019, Kallus and Mao, 2020].

In Section 6 we further relax eq. (4) to allow the distributions of covariates X to be different in the two samples. Note that eq. (4) already allows the distributions of potential long-term outcome Y(a) in the experimental and observational data to be different, so that the long-term treatment effect on the experimental population can be different from our target.

Equation (5) in Assumption 3 means that the conditional distributions of $(U, X) \mid A$ on the experimental and observational data have enough overlap, which is also a common assumption in missing data literature [Tsiatis, 2007].

3.2 Three Groups of Short-term Outcomes

To address general persistent confounding, we need some additional structure on the short-term outcomes. In this paper, we consider leveraging multiple, sequential short-term outcomes. In particular, we consider a partitioning of the short-term outcomes into three groups sorted in a temporal order, writing the potential short-term outcomes as $S(a) = (S_1(a), S_2(a), S_3(a)) \in S_1 \times S_2 \times S_3$ and their observed counterparts as $S = (S_1, S_2, S_3)$. Given this partitioning, we assume the following conditional independence structure for the potential short-term and long-term outcomes.

Assumption 4 (Sequential Outcomes). For $a \in \{0, 1\}$,

$$(Y(a), S_3(a)) \perp S_1(a) \mid S_2(a), U, X, G = O,$$
 (6)

Assumption 4 requires that the effect of the first short-term outcome on the last short-term outcome and the long-term outcome is mediated by the intermediate short-term outcome. Nonetheless, all outcomes can be related by unobserved confounders, even in the experimental data, and the treatment can affect all outcomes both directly and indirectly. This captures the sequential structure of the short-term and long-term outcomes (see Figure 3 for an example). For example, it holds when the potential outcomes follow autoregressive structural equations of suitable orders (see Example 2 below for a simple instance). Moreover, it holds when the potential outcomes $S_1(a), S_2(a), S_3(a), Y(a)$ follow a Markov process, conditional on U, X. Markov models are widely used in social sciences such as for modeling the dynamics of labor markets [Poterba and Summers, 1986, Mohapatra et al., 2007] as well as in medical sciences for modeling chronic-disease progression [Marshall and Jones, 1995, Liu et al., 2013, Kay, 1986, Liu et al., 2013]. Note that previous work

with multiple short-term outcomes usually views them as a single vector of short-term outcomes, and does not put any structure on them [e.g., Athey et al., 2019, Kallus and Mao, 2020]. In contrast, assuming a sequential internal structure among these (Assumption 4) allows us to address the challenge of persistent confounding, as we demonstrate in the next section.

We further assume that short-term outcomes (S_1, S_3) are sufficiently affected by the unobserved confounders U, formalized in the following completeness conditions.

Assumption 5 (Completeness Conditions). For any $s_2 \in \mathcal{S}_2$, $a \in \{0,1\}$, $x \in \mathcal{X}$,

- 1. If $\mathbb{E}[g(U) \mid S_3, S_2 = s_2, A = a, X = x, G = O] = 0$ holds almost surely, then g(U) = 0 almost surely.
- 2. If $\mathbb{E}[g(U) \mid S_1, S_2 = s_2, A = a, X = x, G = O] = 0$ holds almost surely, then g(U) = 0 almost surely.

These completeness conditions require that the short-term outcomes (S_1, S_3) are strongly dependent with the unobserved confounders, and they have sufficient variability relative to the unobserved confounders U. Under these conditions, (S_1, S_3) can be viewed as strong proxy variables¹ for the unobserved confounders U. Completeness assumptions have been used extensively in recent literature on proximal causal inference [Miao et al., 2016, Shi et al., 2020, Miao and Tchetgen, 2018, Cui et al., 2020, Kallus et al., 2021]. However, these literature require proxy variables that are not causally affected by the treatment, termed negative controls. In contrast, here both of (S_1, S_3) can be affected by the treatment and thus do not directly fit into this previous literature.

While in the main text we simply consider a single set of unobserved confounders U that affect everything, in Appendix E.2 we further dissect persistent confounders into groups of variables and show that some unobserved confounders can be ignored and simply excluded from U, relaxing some of the above assumptions.

4 Identification

In this section, we establish three novel identification strategies for the average long-term treatment effect in presence of general persistent confounding.

4.1 Identification via Outcome Bridge Function

We first introduce the concept of an outcome bridge function, which will play an important role in our first identification strategy.

Assumption 6 (Outcome Bridge Function). There exists an outcome bridge function $h_0 : S_3 \times S_2 \times A \times X \to \mathbb{R}$ defined as follows:

$$\mathbb{E}[Y \mid S_2, A, U, X, G = O] = \mathbb{E}[h_0(S_3, S_2, A, X) \mid S_2, A, U, X, G = O]. \tag{7}$$

According to Equation (7), an outcome bridge function h_0 gives a transformation of short-term outcomes (S_3, S_2) , treatment A, and covariates X, such that the confounding effects of the unmeasured variables U on this transformation can reproduce those on the long-term outcome Y. So we can expect outcome bridge functions to be useful in tackling unmeasured confounding.

¹Note that we do not require S_2 to be strong proxy variables. Instead, we only require S_2 to block the path between S_1 and S_3 as depicted in Figure 3, so that Assumption 4 is plausible.

In general nonparametric settings, Assumption 6 holds as a consequence of Assumption 5 condition 1 and some additional technical conditions. See Appendix C for details. In some special cases detailed below, we can both directly guarantee Assumption 6 and describe the functional form of outcome bridge functions.

Example 1 (Discrete Setting). Suppose that $S_1 = S_2 = S_3 = \{s_{(j)} : j = 1, \dots, M_s\}$ and $\mathcal{U} = \{u_{(k)} : k = 1, \dots, M_u\}$. For any $s_2 \in S_2$, $a \in \mathcal{A}$, $x \in \mathcal{X}$, let $\mathbb{E}[Y \mid s_2, a, \mathbf{U}, x] \in \mathbb{R}^{M_u}$ denote the vector whose kth element is $\mathbb{E}[Y \mid S_2 = s_2, A = a, U = u_{(k)}, X = x, G = O]$ and $P(\mathbf{S}_3 \mid s_2, a, \mathbf{U}, x) \in \mathbb{R}^{M_s \times M_u}$ the matrix whose (j, k)th element is $\mathbb{P}(S_3 = s_{(j)} \mid S_2 = s_2, A = a, U = u_{(k)}, X = x, G = O)$. The existence of an outcome bridge function in Assumption 6 is equivalent to the existence of a solution $z \in \mathbb{R}^{M_s}$ to the following linear equation system for any $s_2 \in S_2$, $a \in \mathcal{A}$, $x \in \mathcal{X}$:

$$P(\mathbf{S}_3 \mid s_2, a, \mathbf{U}, x)^{\mathsf{T}} z = \mathbb{E}\left[Y \mid s_2, a, \mathbf{U}, x\right]$$
(8)

A sufficient condition for the existence of solutions to Equation (8) is that the matrix $P(\mathbf{S}_3 \mid s_2, a, \mathbf{U}, x)$ has a full column rank for any $s_2 \in \mathcal{S}_2, a \in \mathcal{A}, x \in \mathcal{X}$. This full column rank condition means that S_3 are strongly dependent with U and it requires that the number of possible values of S_3 (i.e., M_s) is no smaller than the number of possible values of U (i.e., M_u). In this example, the full column rank sufficient condition is equivalent to the completeness condition in Assumption 5 condition 1.

Example 2 (Linear Model). Suppose that (Y, S_3, S_2, S_1) are generated from the following linear structural equation system:

$$Y = \tau_y A + \alpha_y^{\top} S_3 + \beta_y^{\top} X + \gamma_y^{\top} U + \epsilon_y,$$

$$S_j = \tau_j A + \alpha_j S_{j-1} + \beta_j X + \gamma_j U + \epsilon_j, \ j \in \{3, 2\}$$

$$S_1 = \tau_1 A + \beta_1 X + \gamma_1 U + \epsilon_1,$$

where $\tau_y, (\tau_j, \alpha_y, \beta_y, \gamma_y), (\alpha_j, \beta_j, \gamma_j)$ are scalars, vectors, and matrices of conformable sizes, respectively, and ϵ_y, ϵ_j are independent mean-zero noise terms such that $\epsilon_y \perp (S, A, U, X)$ and $\epsilon_j \perp (S_{j-1}, \ldots, S_1, A, U, X)$. Assumption 6 holds if there exists a solution ω to the linear equation $\gamma_3^{\top} \omega = \gamma_y$, since for any such ω , it can be easily shown that a valid outcome bridge function is

$$h_0(s_3, s_2, a, x) = \theta_3^\top s_3 + \theta_2^\top s_2 + \theta_1 a + \theta_0^\top x,$$

where $\theta_3 = \omega + \alpha_y$, $\theta_2 = -\alpha_3^{\mathsf{T}} \omega$, $\theta_1 = \tau_y - \tau_3^{\mathsf{T}} \omega$, $\theta_0 = \beta_y - \beta_3^{\mathsf{T}} \omega$. Therefore, a sufficient condition for the existence of outcome bridge functions is that γ_3 has a full column rank. This full-column-rank condition again means that S_3 is sufficiently informative for the unobserved confounders U.

Note that outcome bridge functions in Equation (7) are defined in terms of unobserved confounders, so we cannot directly use this definition to learn outcome bridge functions from observed data. In the following lemma, we give an alternative characterization of outcome bridge functions, only in terms of distributions of observed data.

Lemma 1. Under Assumptions 1 to 4, the completeness condition in Assumption 5 condition 2 and Assumption 6, any function h_0 that satisfies

$$\mathbb{E}[Y \mid S_2, S_1, A, X, G = O] = \mathbb{E}[h_0(S_3, S_2, A, X) \mid S_2, S_1, A, X, G = O]$$
(9)

is also a valid outcome bridge function in the sense of Equation (7).

In Lemma 1, we assume the completeness condition in Assumption 5 condition 2, which requires the short-term outcomes S_1 to be informative enough for the unobserved confounders U. Under this additional assumption, outcome bridge functions can be equivalently characterized by the conditional moment equation in Equation (9). Note that Equation (9) simply replaces the unobserved confounders U in Equation (7) by the observed short-term outcomes S_1 . The resulting conditional moment equation only depends on observed variables.

We finally establish the identification of the average long-term treatment effect in the following theorem.

Theorem 1. Under the conditions of Lemma 1, the average long-term treatment effect is identifiable: for any function h_0 satisfying Equation (9), at least one of which exists, we have

$$\tau = \mathbb{E}\left[h_0(S_3, S_2, A, X) \mid A = 1, G = E\right] - \mathbb{E}\left[h_0(S_3, S_2, A, X) \mid A = 0, G = E\right]. \tag{10}$$

Theorem 1 states that the average long-term treatment effect can be recovered by marginalizing *any* outcome bridge function (which is defined on the observational data distribution) over the experimental data distribution. This shows how observational and experimental data can be combined together to identify the long-term treatment effect.

Remark 1 (Connection to Athey et al. [2020]). The proposed identification strategy in Equation (10) can be viewed as a generalization of that in Athey et al. [2020]. When there only exist short-term confounders, Athey et al. [2020] shows that we only need a single group of short-term outcomes. We can let $S_1 = S_3 = \emptyset$ and $S = S_2$, then $h_0(S_2, A, X) = \mathbb{E}[Y \mid S, A, X, G = O]$ is the unique solution to Equation (9), and it can be plugged into Equation (10) to identify the average long-term treatment effect. This recovers the identification strategy in Theorem 1 of Athey et al. [2020] when specialized to the case of Assumption 3 (Corollary 1 in Appendix D.1 recovers it in the general case; see discussions therein). Of course, when persistent confounding is present this identification fails and instead Theorem 1 provides a more general identification strategy that can leverage structure in the surrogates to handle persistent confounders.

4.2 Identification via Selection Bridge Function

The second identification strategy involves an alternative function called selection bridge function.

Assumption 7 (Selection Bridge Function). There exists a selection bridge function $q_0 : \mathcal{S}_2 \times \mathcal{S}_1 \times \mathcal{A} \times \mathcal{X} \to \mathbb{R}$ defined as follows:

$$\frac{p(S_2, U, X \mid A, G = E)}{p(S_2, U, X \mid A, G = O)} = \mathbb{E}\left[q_0(S_2, S_1, A, X) \mid S_2, A, U, X, G = O\right]. \tag{11}$$

According to Equation (11), a selection bridge function q_0 gives a transformation of short-term outcomes (S_2, S_1) , treatment A, and covariates X, which can adjust for distributional differences between the experimental and observational data. In Appendix G.1 Lemma 7, we prove that under assumption 3, the density ratio in left hand side of Equation (11) is almost surely finite, so Equation (11) is well-defined.

In general nonparametric models, the existence of a selection bridge function can be ensured by the completeness condition in Assumption 5 condition 2 and some additional technical conditions. See Appendix C for details. This means that a selection bridge function exists when the short-term outcomes S_1 are sufficiently informative for the unobserved confounders U. We can also derive more specialized existence conditions for Examples 1 and 2 (see Appendix B).

Again, selection bridge functions in Equation (11) are defined in terms of unobserved confounders. Below, we derive alternative characterizations in terms of distributions of observed variables.

Lemma 2. Under assumptions 1 to 4, the completeness condition in Assumption 5 condition 1, and Assumption 7, any function q_0 that satisfies

$$\frac{p(S_3, S_2, X \mid A, G = E)}{p(S_3, S_2, X \mid A, G = O)} = \mathbb{E}\left[q_0(S_2, S_1, A, X) \mid S_3, S_2, A, X, G = O\right]$$
(12)

is also a valid selection bridge function in the sense of Equation (11).

In Lemma 2, we assume the completeness condition in Assumption 5 condition 1, which requires the short-term outcomes S_3 to be informative enough for the unobserved confounders U. Under this additional assumption, selection bridge functions can be equivalently characterized by the conditional moment equation in Equation (12), which involves only observed variables. Equation (12) is a direct analogue to Equation (11), replacing U in Equation (11) by S_3 in Equation (12). We can also equivalently express Equation (12) as follows

$$\mathbb{E}\left[\mathbb{I}\left[G = O\right] \left(\frac{\mathbb{P}\left(G = E \mid A\right)}{\mathbb{P}\left(G = O \mid A\right)} q_0(S_2, S_1, A, X) + 1\right) \mid S_2, S_1, A, X\right] = 1.$$
(13)

Equation (13) is a more convenient formulation for estimation as it does not involve any conditional density function.

Theorem 2. Under conditions in Lemma 2, the average long-term treatment effect is identifiable: for any function q_0 that satisfies Equation (12) or Equation (13), at least one of which exists, we have

$$\tau = \mathbb{E}\left[q_0(S_2, S_1, A, X)Y \mid A = 1, G = O\right] - \mathbb{E}\left[q_0(S_2, S_1, A, X)Y \mid A = 0, G = O\right]. \tag{14}$$

Theorem 2 states that the average long-term treatment effect can be also identified by *any* selection bridge function. This provides an alternative to the identification strategy based on outcome bridge functions in Theorem 1.

Remark 2 (Comparison with Proximal Causal Inference). As we discussed in Section 2.3, our identification strategies are related to identification in the proximal causal inference literature. Indeed, we also take a proxy-variable perspective, viewing short-term outcomes (S_1, S_3) as proxy variables for the unobserved confounders U. Moreover, the characterization for outcome bridge function h_0 given in Equation (9) has an analogue in Miao and Tchetgen [2018].

Nevertheless, our setting is substantially different from the existing proximal causal inference literature. The short-term outcomes (S_1, S_3) are both affected by the treatment, so they do not satisfy the proxy conditions in Miao et al. [2016]. Our identification strategies also feature a novel use of the experimental data. This is crucial in our setting, whereas proximal causal inference focuses on observational data only. Notably, our identification strategy in Theorem 2 relies on a new selection bridge function. This bridge function, as defined in eq. (12), is specialized to our data combination setting, without direct analogue in the existing proximal causal inference literature except the concurrent work Ghassami et al. [2022].

Remark 3 (Assumptions 5 to 7 and the Conditioning on S_2). In Assumption 5 we assume two completeness conditions and in Assumptions 6 and 7, we assume the exsitence of outcome and

selection bridge functions. These conditions roughly require S_1 , S_3 to be strongly dependent with the unobserved confounders U after accounting for S_2 , A and X. Since S_2 also tend to be dependent with U, conditioning on S_2 may explain away part of the dependence between S_1 , S_3 and U. Thus Assumptions 5 to 7 may be at risk if S_2 include very rich short-term outcomes and capture a very large amount of variations in U. They are more plausible as S_1 , S_3 include richer informative short-term outcomes relative to S_2 .

4.3 Doubly Robust Identification

In Sections 4.1 and 4.2, we present two different identification strategies, based on outcome bridge functions and selection bridge functions, respectively. We now combine them into a doubly robust identification strategy.

Theorem 3. Fix functions $h: S_3 \times S_2 \times A \times X \to \mathbb{R}$ and $q: S_2 \times S_1 \times A \times X \to \mathbb{R}$. If either conditions in Theorem 1 hold and $h = h_0$ satisfies eq. (9), or conditions in Theorem 2 hold and $q = q_0$ satisfies eq. (12) or eq. (13), then the average long-term treatment effect is identified as:

$$\tau = \sum_{a \in \{0,1\}} (-1)^{1-a} \mathbb{E} \left[h(S_3, S_2, A, X) \mid A = a, G = E \right]$$

$$+ \sum_{a \in \{0,1\}} (-1)^{1-a} \mathbb{E} \left[q(S_2, S_1, A, X) (Y - h(S_3, S_2, A, X)) \mid A = a, G = O \right].$$

$$(15)$$

Theorem 3 shows that Equation (15) identifies the average long-term treatment effect when it uses either a valid outcome bridge function or a valid selection bridge function. But it does not need both bridge functions to be valid. This is why it is called doubly robust.

5 Estimation and Inference

In this section, we provide three different estimators for the average long-term treat effect, corresponding to the three different identification strategies in Section 4 respectively. This involves combining two samples, so we assume that as $n \to \infty$, $n_E/n_O \to \lambda$ where $0 < \lambda < \infty$. This is a common assumption in the data combination literature [e.g., Angrist and Krueger, 1992, Graham et al., 2016].

In order to estimate the average long-term treatment effect, we need to first estimate the outcome and/or selection bridge functions. Estimating these bridge functions amounts to solving the conditional moment equations in Equations (9) and (13) based on a finite sample of data, which corresponds to an ill-posed inverse problem [Carrasco et al., 2007]. A variety of estimation strategies can be used for this task, which we review in Remark 4 below. For now, we consider any generic bridge function estimators, which may be any from those reviewed in Remark 4, and discuss different ways to use these to construct the long-term treatment effect estimator.

Below, we define three different estimators for the counterfactual mean parameter $\mu(a)$, $a \in \mathcal{A}$. They all use the cross-fitting technique when constructing bridge function estimators. This technique has been widely used to accommodate complex nuisance function estimators while preserving strong asymptotic guarantees [e.g., Chernozhukov et al., 2019, Zheng and Laan, 2011].

Definition 1 (Cross-fitted Counterfactual Mean Estimator). Fix $a \in \mathcal{A}$ and an integer K > 2.

1. Randomly split the observational data \mathcal{D}_O into K (approximately) even folds, denoted as $\mathcal{D}_{O,1},\ldots,\mathcal{D}_{O,K}$, respectively.

- 2. For k = 1, ..., K, use all observational data other than the kth fold, i.e., $\mathcal{D}_{O,-k} := \bigcup_{j \neq k} \mathcal{D}_{O,j}$, to construct the outcome bridge function estimator based on Equation (9) and/or the selection bridge function estimator based on Equation (13). Denote them as $\hat{h}_k(S_3, S_2, A, X)$ and $\hat{q}_k(S_2, S_1, A, X)$, respectively.
- 3. Use any of the following counterfactual mean estimators:

$$\begin{split} \hat{\mu}_{OTC}(a) &= \frac{1}{K} \sum_{k=1}^{K} \left[\frac{1}{n_{E}^{(a)}} \sum_{i \in \mathcal{D}_{E}} \mathbb{I} \left[A_{i} = a \right] \hat{h}_{k}(S_{3,i}, S_{2,i}, A_{i}, X_{i}) \right], \\ \hat{\mu}_{SEL}(a) &= \frac{1}{K} \sum_{k=1}^{K} \left[\frac{1}{n_{O,k}^{(a)}} \sum_{i \in \mathcal{D}_{O,k}} \mathbb{I} \left[A_{i} = a \right] \hat{q}_{k}(S_{2,i}, S_{1,i}, A_{i}, X_{i}) Y_{i} \right], \\ \hat{\mu}_{DR}(a) &= \frac{1}{K} \sum_{k=1}^{K} \left[\frac{1}{n_{E}^{(a)}} \sum_{i \in \mathcal{D}_{E}} \mathbb{I} \left[A_{i} = a \right] \hat{h}_{k}(S_{3,i}, S_{2,i}, A_{i}, X_{i}) \right] \\ &+ \frac{1}{K} \sum_{k=1}^{K} \left[\frac{1}{n_{O,k}^{(a)}} \sum_{i \in \mathcal{D}_{O,k}} \mathbb{I} \left[A_{i} = a \right] \hat{q}_{k}(S_{2,i}, S_{1,i}, A_{i}, X_{i}) \left(Y_{i} - \hat{h}_{k}(S_{3,i}, S_{2,i}, A_{i}, X_{i}) \right) \right], \end{split}$$

where $n_E^{(a)} = \sum_{i \in \mathcal{D}_E} \mathbb{I}[A_i = a]$ and $n_{O,k}^{(a)} = \sum_{i \in \mathcal{D}_{O,k}} \mathbb{I}[A_i = a]$ are the numbers of units with treatment level a in the experimental data \mathcal{D}_E and the k-th fold of observational data $\mathcal{D}_{O,k}$, respectively.

Based on the counterfactual mean estimators in Definition 1, we can construct average long-term treatment effect estimators:

$$\hat{\tau}_{\text{OTC}} = \hat{\mu}_{\text{OTC}}(1) - \hat{\mu}_{\text{OTC}}(0), \quad \hat{\tau}_{\text{SEL}} = \hat{\mu}_{\text{SEL}}(1) - \hat{\mu}_{\text{SEL}}(0), \quad \hat{\tau}_{\text{DR}} = \hat{\mu}_{\text{DR}}(1) - \hat{\mu}_{\text{DR}}(0).$$

To analyze the asymptotic properties of these treatment effect estimators, we need to impose some high level conditions on the estimation errors of the bridge function estimators. Since these estimators solve ill-posed conditional moment equations, we quantify their estimation errors in terms of both weak metrics and the strong metrics, as this is a common practice in the literature [e.g., Chen and Pouzo, 2012, Dikkala et al., 2020]. In particular, we define a projection operator T and its adjoint operator T^* given by $[Th](S_2, S_1, A, X) = \mathbb{E}[h(S_3, S_2, A, X) \mid S_2, S_1, A, X, G = O]$ and $[T^*q](S_3, S_2, A, X) = \mathbb{E}[q(S_2, S_1, A, X) \mid S_3, S_2, A, X, G = O]$. For a given outcome bridge function estimator \hat{h} and a given selection bridge function estimator \hat{q} , we can quantify their estimation errors relative to h and q in terms of the weak metrics $||T(\hat{h}-h)||_{\mathcal{L}_2(\mathbb{P})}$ and $||T^*(\hat{q}-q)||_{\mathcal{L}_2(\mathbb{P})}$ respectively. We can also quantify their estimation errors in terms of the strong metrics $||\hat{h}-h||_{\mathcal{L}_2(\mathbb{P})}$ and $||\hat{q}-q||_{\mathcal{L}_2(\mathbb{P})}$ respectively. The strong-metric errors can be much larger (even infinitely larger) than the corresponding weak-metric errors due to ill-posedness of the conditional moment equations.

Assumption 8 (Error Rates of Bridge Function Estimators). 1. There exist $\tilde{h} \in \mathcal{S}_3 \times \mathcal{S}_2 \times \mathcal{A} \times \mathcal{X} \to \mathbb{R}$ and sequences $\delta_{h,n} \to 0$ and $\rho_{h,n} \to 0$ such that

$$||T(\hat{h}_k - \tilde{h})||_{\mathcal{L}_2(\mathbb{P})} = O_{\mathbb{P}}(\delta_{h,n}), ||\hat{h}_k - \tilde{h}||_{\mathcal{L}_2(\mathbb{P})} = O_{\mathbb{P}}(\rho_{h,n}), \forall k \in \{1, \dots, K\}.$$

2. There exist $\tilde{q} \in \mathcal{S}_2 \times \mathcal{S}_1 \times \mathcal{A} \times \mathcal{X} \to \mathbb{R}$ and sequences $\delta_{q,n} \to 0$ and $\rho_{q,n} \to 0$ such that

$$||T^{\star}(\hat{q}_k - \tilde{q})||_{\mathcal{L}_2(\mathbb{P})} = O_{\mathbb{P}}(\delta_{q,n}), \ ||\hat{q}_k - \tilde{q}||_{\mathcal{L}_2(\mathbb{P})} = O_{\mathbb{P}}(\rho_{q,n}), \ \forall k \in \{1, \dots, K\}.$$

Assumption 8 specifies that the outcome bridge function estimator and selection bridge function estimator converge to some limits \tilde{h} and \tilde{q} respectively, in terms of both weak metrics and strong metrics. Note that we do not necessarily require these estimators to be consistent, *i.e.*, we allow $\tilde{h} \neq h_0$ or $\tilde{q} \neq q_0$, as we show in the following theorem.

Theorem 4 (Estimation Consistency). 1. If conditions in Theorem 1 and Assumption 8 condition 1 hold, and $\tilde{h} = h_0$, then $\hat{\tau}_{OTC}$ consistent.

- 2. If conditions in Theorem 2 and Assumption 8 condition 2 hold, and $\tilde{q} = q_0$, then $\hat{\tau}_{SEL}$ is consistent.
- 3. If the conditions in either of the two statements above hold, then $\hat{\tau}_{DR}$ is consistent.

Theorem 4 shows that if the outcome bridge function estimator is consistent (i.e., $\tilde{h} = h_0$), then the corresponding treatment effect estimator $\hat{\tau}_{\text{OTC}}$ is consistent. Similarly, if the selection bridge function estimator is consistent (i.e., $\tilde{q} = q_0$), then the corresponding treatment effect estimator $\hat{\tau}_{\text{SEL}}$ is also consistent. In contrast, the estimator $\hat{\tau}_{\text{DR}}$ is more robust, in that it is consistent if either of the two bridge function estimators is consistent.

Theorem 4 establishes the consistency of treatment effect estimators given only high level conditions on the bridge function estimators, regardless of how they are actually constructed. However, the actual ways to construct bridge function estimators generally do impact the asymptotic distributions of treatment effect estimators $\hat{\tau}_{OTC}$ and $\hat{\tau}_{SEL}$. So we only focus on the asymptotic distribution of estimator $\hat{\tau}_{DR}$, which can be derived even under generic high level conditions.

Theorem 5 (Asymptotic Distribution of Doubly Robust Estimator). Suppose that conditions in both Theorem 4 statement 1 and Theorem 4 statement 2 hold and min $\{\delta_{h,n}\rho_{q,n}, \rho_{h,n}\delta_{q,n}\} = o(n^{-1/2})$. Then as $n \to \infty$,

$$\sqrt{n}(\hat{\tau}_{DR} - \tau) \rightsquigarrow \mathcal{N}(0, \sigma^2),$$

where

$$\sigma^{2} = \frac{1+\lambda}{\lambda} \mathbb{E} \left[\left(\frac{A - \mathbb{P}(A=1 \mid G=E)}{\mathbb{P}(A=1 \mid G=E)} (h_{0}(S_{3}, S_{2}, A, X) - \mu(A)) \right)^{2} \mid G=E \right]$$

$$+ (1+\lambda) \mathbb{E} \left[\left(\frac{A - \mathbb{P}(A=1 \mid G=O)}{\mathbb{P}(A=1 \mid G=O)} q_{0}(S_{2}, S_{1}, A, X) (Y - h_{0}(S_{3}, S_{2}, A, X)) \right)^{2} \mid G=O \right].$$

Theorem 5 shows that if both bridge function estimators are consistent (i.e., $h = h_0$ and $\tilde{q} = q_0$), and the product of their convergence rates in terms of one strong-metric error and one weak-metric error is $o(n^{-1/2})$, then the doubly robust treatment effect estimator $\hat{\tau}_{DR}$ is asymptotically normal with a closed-form asymptotic variance. Note that the rate condition is weaker than requiring the product of two strong-metric error rates to be $o(n^{-1/2})$. We can easily estimate this asymptotic variance by plugging estimates into all unknowns therein:

$$\hat{\sigma}^{2} = \frac{n}{n_{E}K} \sum_{k=1}^{K} \left\{ \frac{1}{n_{E}} \sum_{i \in \mathcal{D}_{E}} \left[\frac{A_{i} - \hat{\pi}_{E}}{\hat{\pi}_{E}} \left(\hat{h}_{k}(S_{3,i}, S_{2,i}, A_{i}, X_{i}) - \hat{\mu}_{DR}(A_{i}) \right) \right]^{2} \right\}$$

$$+ \frac{n}{n_{O}K} \sum_{k=1}^{K} \left\{ \frac{1}{n_{O,k}} \sum_{i \in \mathcal{D}_{O,k}} \left[\frac{A_{i} - \hat{\pi}_{O}}{\hat{\pi}_{O}} \hat{q}_{k}(S_{2,i}, S_{1,i}, A_{i}, X_{i}) \left(Y_{i} - \hat{h}_{k}(S_{3,i}, S_{2,i}, A_{i}, X_{i}) \right) \right]^{2} \right\}, \quad (16)$$

where $\hat{\pi}_E$ and $\hat{\pi}_O$ are sample frequency estimates for $\mathbb{P}(A=1 \mid G=E)$ and $\mathbb{P}(A=1 \mid G=O)$ respectively. Then we can construct confidence intervals based on this estimated asymptotic variance.

Theorem 6 (Confidence Interval). Under the conditions in Theorem 5, the confidence interval

CI =
$$\left[\hat{\tau}_{DR} - \Phi^{-1}(1 - \alpha/2)\hat{\sigma}/\sqrt{n}, \ \hat{\tau}_{DR} + \Phi^{-1}(1 - \alpha/2)\hat{\sigma}/\sqrt{n}\right]$$

satisfies that

$$\mathbb{P}\left(\tau \in \mathrm{CI}\right) \to 1 - \alpha, \quad as \ n \to \infty.$$

In the following theorem, we further show that the asymptotic variance in Theorem 5 actually attains the local semiparametric efficiency lower bound, provided that the bridge functions uniquely exist and an additional regularity condition holds.

Theorem 7 (Asymptotic Efficiency). Let \mathbb{P} be a distribution instance such that Assumptions 6 and 7 hold with unique bridge functions and the corresponding linear operator T defined above Assumption 8 is bijective. Then, the efficiency lower bound for the average long-term treatment effect τ under Assumptions 1 to 4 and 6, locally evaluated at the distribution \mathbb{P} , is equal to σ^2 given in Theorem 5.

Theorem 7 implies that under the asserted assumptions, treatment effect estimator $\hat{\tau}_{DR}$ is asymptotically optimal, in the sense that it achieves the smallest asymptotic variance among all regular and asymptotically linear estimators [Van der Vaart, 2000].

Remark 4 (Bridge Function Estimators). Estimating bridge functions amounts to estimating roots of the conditional moment equations in Equations (9) and (13). This can be implemented by a variety of methods. Examples include Generalized Method of Moments (GMM) [e.g., Miao and Tchetgen, 2018, Cui et al., 2020, Hansen, 1982], sieve methods [e.g., Ai and Chen, 2003, Newey and Powell, 2003, Hall and Horowitz, 2005], kernel density estimators [e.g., Darolles et al., 2010, Hall and Horowitz, 2005], Reproducing Kernel Hilbert Space methods [e.g., Singh et al., 2019, Ghassami et al., 2021b], neural network methods [e.g., Hartford et al., 2017, Bennett et al., 2019], and more generally, adversarial learning methods [e.g., Bennett and Kallus, 2020, Dikkala et al., 2020, Kallus et al., 2021]. We can use any of these to estimate the bridge functions. Some of these also provide theoretical guarantees on the resulting convergence rates (in the sense of Assumption 8). In particular, the weak-metric error rates are readily available in many of these works, but the strong-metric error rates typically need additional restrictions on the ill-posedness of the conditional moment equations [Chen and Pouzo, 2012, Dikkala et al., 2020].

Remark 5 (Non-uniqueness of Bridge Functions). In Theorem 7, we assume that bridge functions uniquely exist, which is not necessarily true in practice. As we discussed in Sections 4.1 and 4.2, bridge functions exist if the short-term outcomes S_1 and S_3 are sufficiently informative for the unobserved confounders. But when they are more informative than necessary, there may exist more than one bridge function. For example, in Example 1, when the matrix $P(\mathbf{S}_3 \mid s_2, a, \mathbf{U}, x)$ has full column rank and S_3 has more values than the unobserved confounders U, Equation (8) admits many solutions z and each of them corresponds to a different outcome bridge function.

The non-uniqueness of bridge functions has important implications for asymptotic properties of treatment effect estimators. Almost all previous results in proximal causal inference assume unique bridge functions when studying statistical inference. One exception is the penalized GMM estimator in Imbens et al. [2021], which leverages penalization to power inference even with non-unique bridge functions. But their approach only applies to parametric estimation of bridge functions. Bennett et al. [2022] proposes methods for inference on functionals of solutions to weakly identified nonparametric conditional moment equations, and consider proximal causal inference with non-unique bridge functions as a canonical example.

6 Extensions

In this section, we extend our previous identification results. We first relax Assumptions 2 and 3 in Section 6.1. Then in Section 6.2 we provide an alternative identification strategy via control functions rather than bridge functions. This approach can identify not only the average long term treatment effect but also the entire distribution of the counterfactual long term outcomes.

6.1 Relaxing Assumptions 2 and 3

We now extend our identification results by relaxing Assumptions 2 and 3. In particular, we relax Assumption 3 by allowing the covariate distribution to be different in the experimental and observational data. This is an important extension because these two types of data are often collected from different environments, where the covariate distributions are likely to be different. For example, because observational data are usually easier to collect and have larger scale than experimental data, the observational covariate distribution may be more representative of the entire population of interest, while experimental data may only correspond to a selective sub-population. Therefore, we consider the following assumption to allow for different covariate distributions in two types of data.

Assumption 9 (External Validity, Modified). Suppose that for any $a \in \{0, 1\}$,

$$(S(a), U) \perp G \mid X, \tag{17}$$

and Equation (5) holds almost surely.

Moreover, we relax Assumption 2 by allowing the treatment assignment in the experimental data to depend on covariates X, instead of being completely at random. This permits us to also accommodate stratified randomized designs for the experimental data.

Assumption 10 (Experimental Data, Modified). Suppose that for any $a \in \{0, 1\}$,

$$(Y(a), S(a), U) \perp A \mid X, G = E, \tag{18}$$

and $0 < \mathbb{P}(A = 1 \mid X, G = E) < 1$ almost surely.

Below we extend the doubly robust identification strategy in Theorem 3, which shows that the long-term average treatment effect is still identifiable under the weaker Assumptions 9 and 10.

Theorem 8. Fix functions $h: S_3 \times S_2 \times A \times X \to \mathbb{R}$, $q: S_2 \times S_1 \times A \times X \to \mathbb{R}$, and denote $\bar{h}_E(a,x) = \mathbb{E}[h(S_3,S_2,A,X) \mid A=a,X=x,G=E]$. Suppose Assumptions 1, 4, 9 and 10 hold, and either of the following two conditions holds:

- 1. The completeness condition in Assumption 5 condition 2 and Assumption 6 hold, and $h = h_0$ satisfies Equation (9);
- 2. The completeness condition in Assumption 5 condition 1 and Assumption 7 hold, and $q = q_0$ satisfies Equation (12) or Equation (13).

Then the average long-term treatment effect is identified as:

$$\tau = \sum_{a \in \{0,1\}} (-1)^{1-a} \left\{ \mathbb{E} \left[\bar{h}_{E}(a,X) \mid G = O \right] + \mathbb{E} \left[\frac{\mathbb{P} \left(G = E \right) \mathbb{P} \left(G = O \mid X \right)}{\mathbb{P} \left(G = O \right) \mathbb{P} \left(G = E \mid X \right)} \frac{\mathbb{I} \left[A = a \right]}{\mathbb{P} \left(A = a \mid X, G = E \right)} \left(h(S_{3}, S_{2}, A, X) - \bar{h}_{E}(A, X) \right) \mid G = E \right] + \mathbb{E} \left[\frac{\mathbb{P} \left(G = E \mid A = a \right) \mathbb{P} \left(G = O \mid X \right)}{\mathbb{P} \left(G = O \mid A = a \right) \mathbb{P} \left(G = E \mid X \right)} \frac{\mathbb{I} \left[A = a \right]}{\mathbb{P} \left(A = a \mid X, G = E \right)} q\left(S_{2}, S_{1}, A, X \right) \left(Y - h\left(S_{3}, S_{2}, A, X \right) \right) \mid G = O \right] \right\}$$

Theorem 8 shows that even under the weaker Assumptions 9 and 10, outcome and selection bridge functions can still be used to identify the average long-term treatment effect. This again has the doubly robust property in that it only requires one of the bridge functions to be correct rather than both. Compared to Theorem 3, Theorem 8 additionally incorporates the ratio $\mathbb{P}(G = O \mid X)/\mathbb{P}(G = E \mid X)$ to adjust for the discrepancy in covariate distribution of the two types of data (Assumption 9). It also uses the propensity score $\mathbb{P}(A = a \mid X, G = E)$ to account for the dependence of treatment A on covariates X in the experimental data (Assumption 10).

In Appendix D.1, we further show that by setting q = 0, $h = h_0$ or h = 0, $q = q_0$ in Theorem 8, we can obtain direct analogues of Theorems 1 and 2 that involve only a single bridge function. In Appendix D.2, we prove that the estimating equation based on the doubly robust identification strategy in Theorem 8 satisfies the *Neyman orthogonality* property [Chernozhukov et al., 2019], and show that the resulting treatment effect estimator has appealing asymptotic properties and is amenable to inference.

In Appendix E, we present some additional extensions. In Appendix E.1, we extend our identification strategies to the setting where pre-treatment outcomes are available. In Appendix E.2, we show that it is possible to relax completeness conditions in Assumption 5 so that short-term outcomes need only be rich enough to capture only some of the unobserved confounders rather than all of them.

Remark 6 (Connection to Ghassami et al. [2022]). The doubly robust identification strategy in Theorem 8 and its close variants based on only a single bridge function (see Corollary 1 in Appendix D.1) have close analogues in the concurrent and dependent work Ghassami et al. [2022]. Specifically, the proximal data fusion identification strategies in Ghassami et al. [2022] use a set of short-term outcomes M and an additional set of proxies Z that satisfy $Z \perp (M,Y) \mid A, X, U, G = O$. We note that under our sequential outcome condition in Assumptions 1 and 4, we have $S_1 \perp (S_3,Y) \mid S_2,A,X,U,G=O$. The identification strategies in Ghassami et al. [2022], when their Z and M are replaced by S_1 and S_3 respectively and S_2 is additionally conditioned on everywhere, are actually equivalent to our identification strategies. Despite the close relations to Ghassami et al. [2022], our paper uniquely shows that short-term outcomes alone suffice for addressing unmeasured confounding and enables this by assuming a novel sequential outcome condition. As a result, we do not need to look for any external proxy, which may be often unavailable in practice. Instead, we can focus on only short-term outcomes like existing literature [e.g., Athey et al., 2019, 2020, Kallus and Mao, 2020]. In addition, we also provide many additional extensions that have no analogues in Ghassami et al. [2022]. See Section 2.2 for a summary.

6.2 A Control Function Approach

In previous parts, we focus on identifying the long-term treatment effect using bridge functions. In this part, we provide an alternative identification approach based on a control function. Control functions are special variables constructed from existing variables that can help correct for confounding bias by conditioning on them [Wooldridge, 2015]. Control functions are often constructed from instrumental variables [e.g., Blundell and Powell, 2003, Imbens and Newey, 2009, Florens et al., 2008], but Nagasawa [2018] recently proposes control functions based on proxy variables under assumptions similar to those in the proximal causal inference literature (see the review in Section 2.3). We extend this approach to our setting of long term causal inference. This extension is not straightforward, noting that the assumptions of proximal causal inference are not exactly satisfied in our setting (see Remark 2).

Specifically, we will show that the stochastic process $\mathcal{V} := \{p(s_3 \mid S_2, S_1, A, X, G = O) : s_3 \in \mathcal{S}_3\}$ can be used as a valid control function to identify the long term treatment effect. Here we consider

identifying the expectation of an arbitrary transformation of the counterfactual long term outcome, a more general parameter than the average effect parameter considered so far.

Theorem 9. Suppose Assumptions 1, 4, 9 and 10 and the completeness condition in Assumption 5 condition 1 hold. Moreover, assume for $a \in \{0,1\}$, the support of \mathcal{V} given $S_2, A = a, X, G = O$ is identical to the support of \mathcal{V} given $S_2, X, G = O$. Then for any function $r : \mathcal{V} \mapsto \mathbb{R}$ and $a \in \{0,1\}$,

$$\mathbb{E}\left[r(Y(a)) \mid G = O\right] = \mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[r(Y) \mid \mathcal{V}, S_2, A = a, X, G = O\right] \mid A = a, X, G = E\right] \mid G = O\right]. \tag{19}$$

Besides the running Assumptions 1, 4, 9 and 10, Theorem 9 also imposes the completeness condition in Assumption 5 condition 1 and a common support condition. This completeness condition requires S_3 to be sufficiently informative for the unobserved confounders U, after taking into account other relevant variables. The common support condition enables us to vary A while holding constant the control function \mathcal{V} after conditioning on $S_2, X, G = O$. It is equivalent to the overlap condition that $0 < \mathbb{P}(A = 1 \mid \mathcal{V}, S_2, X, G = O) < 1$ almost surely. This condition is possible only when S_1 can induce sufficient extra variations in \mathcal{V} , or alternatively, when S_1 has a large support and it is sufficiently informative for U [Nagasawa, 2018]. Common support conditions like this are prevalent in the control function literature. See Nagasawa [2018], Imbens and Newey [2009] for more discussions and justifications.

We note that the identification formula in Equation (19) can be used to identify not only the average effect, but also the entire distribution of the counterfactual long term outcome Y(a). This can be achieved by applying Equation (19) to the indicator function $r(\cdot) = \mathbb{I}\left[\cdot \leq y\right]$ for all $y \in \mathcal{Y}$. Actually, under the condition 2 in Theorem 8, we can also use a selection bridge function to identify the entire distribution of Y(a) (see Corollary 3 in Appendix D.1). The condition 2 in Theorem 8 (i.e., the existence of a selection bridge function and the completeness condition in Assumption 5 condition 2) has similar qualitative implications as the completeness condition and common support condition in Theorem 9: they require both S_1 and S_3 to be sufficiently strong proxies for the unobserved confounders U. However, these two set of conditions are in general not directly comparable. See Nagasawa [2018] for more discussions on the connections between conditions in the control function approach and conditions in the bridge function approach.

Finally, we remark that estimating the target parameter based on the identification formula in Theorem 9 may be challenging. On the one hand, the common support condition may fail in practical applications [Chernozhukov et al., 2020]. We may follow Nagasawa [2018], Newey and Stouli [2021] and impose additional (semi)-parametric restrictions on the function $\mathbb{E}\left[r(Y)\mid\mathcal{V},S_2,A=a,X,G=O\right]$. These assumptions can allow for model extrapolation across different values of \mathcal{V} , thereby relaxing the common support condition. Another possibility is to derive partial identification bounds when the common support function is violated. On the other hand, the control function approach requires controlling for an infinitely dimensional stochastic process \mathcal{V} , which cannot be implemented exactly in practice. Nagasawa [2018] proposes a dimension reduction technique for the estimation of causal effects in the proximal causal inference setting. Similar techniques may be also useful in our setting. We leave the development of practical estimation methods based on the control function for the future study.

7 Numerical Studies

7.1 Real data analysis

In this section, we illustrate the performance of our proposed estimators using data for the Greater Avenues to Independence (GAIN) job training program in California. GAIN is a job assistance

program from the late 1980s designed to help low-income population. To evaluate its real impacts on employment, MDRC conducted a randomized experiment in 6 California counties. We use the dataset analyzed in Athey et al. [2019] and focus on two counties: San Diego and Riverside. For each experiment participant, the dataset records a binary treatment variable indicating enrollment in the GAIN program, quarterly job employment information after treatment assignment, and other covariate information (e.g., age, education, marriage). See Hotz et al. [2006], Athey et al. [2019] for more information about the GAIN program.

In our numerical studies, we consider the San Diego data as our experimental dataset \mathcal{D}_E , and construct an observational dataset \mathcal{D}_O based on the Riverside data via a biased subsampling described below. Then we apply our proposed estimators $\hat{\tau}_{\text{OTC}}$, $\hat{\tau}_{\text{SEL}}$, and $\hat{\tau}_{\text{DR}}$ to estimate the average treatment effect of the GAIN program on the long-term employment. Since the original data are from randomized experiments, we consider the average treatment effect thereof as the "ground truth" and use it to evaluate the errors of different estimators.

7.1.1 Data Preparation

For the experimental dataset, we directly use data from San Diego, which include $n_E^{(1)} = 6978$ people in the treatment group and $n_E^{(0)} = 1154$ people in the control group. For the observational dataset, we subsample from the Riverside data, which originally include $N_1 = 4405$ people in the treatment group and $N_0 = 1040$ people in the control group.

We randomly subsample units from the Riverside data according to a sampling probability function $\pi(A, U) \in (0, 1)$, where $A \in \{0, 1\}$ is the treatment assignment and $U \in \{0, 1, 2, 3\}$ is the highest education level ("0" means below 9-th grade, "1" means 9-th to 11-th grade, "2" means 12-th grade, and "3" means above 12-th grade). This creates dependence between the treatment assignment and the education level for the units subsampled into \mathcal{D}_O . We choose education because it is quite likely to have *persistent* effects on participants' potential employment in all quarters following the treatment. Then we drop the education level data from \mathcal{D}_O (and also \mathcal{D}_E). As a result, the education level becomes a plausible persistent unmeasured confounder in \mathcal{D}_O .

To quantify the strength of unmeasured confounding in \mathcal{D}_O , we index the sampling probability function $\pi(A, U)$ by a positive parameter η . We set the sampling probability for control units as $\pi(0, U) = \max\{1 - \eta U/3, 0.2\}$ and the sampling probability for treated units as $\pi(1, U)$ that satisfies the following equation:

$$\frac{N_0}{N_0 + N_1} \pi(0, U) + \frac{N_1}{N_0 + N_1} \pi(1, U) = \frac{N_1}{N_1 + N_0} + \frac{N_0}{N_1 + N_0} \max\{1 - \eta, 0.2\}.$$

It is easy to show that as η grows, the discrepancy between $\pi(0, U)$ and $\pi(1, U)$ also grows. This implies stronger dependence between U and A in the observational dataset \mathcal{D}_O , thus stronger unmeasured confounding. In Appendix F Proposition 3, we prove that with this choice of $\pi(1, U)$, the subsampling procedure does not shift the distribution of education level U, so that it does not violate Assumption 3. Moreover, the subsampling procedure does not influence Assumptions 1 and 2 since the sampling probability function only depends on A, U.

In our numerical studies, we consider the short-term of our comes (S_1, S_2, S_3) as the employment status in the first two quarters, in the third and fourth quarters, and in the fifth and sixth quarters after the treatment respectively. We consider the long-term outcome Y as the 20-th quarter employment. These are all binary variables indicating whether the participants are employed in the corresponding quarters after the treatment assignments.

		$\hat{ au}_{ ext{OTC}}$			$\hat{ au}_{ ext{SEL}}$				$\hat{ au}_{ m DR}$				Athey et al.		Naive	
$\overline{\eta}$		0	.33	.67	1	0	.33	.67	1	0	.33	.67	1	NR	CV	
0	MAE	67	89	84	82	81	81	80	80	71	95	90	88	11	17	0.053
	Med	67	89	84	82	81	81	80	80	71	95	90	88	11	17	0.053
0.2	MAE	18	84	80	78	79	79	79	79	24	89	86	85	19	15	0.059
	Med	61	84	80	78	79	79	79	79	65	90	87	85	18	15	0.059
0.4	MAE	17	79	75	74	76	76	76	76	23	84	82	80	25	13	0.067
	Med	62	79	76	74	77	76	76	76	65	85	83	82	25	13	0.067
0.6	MAE	10	73	70	69	72	72	72	72	17	79	77	76	31	11	0.076
0.0	Med	60	74	71	69	73	73	72	72	63	80	78	77	31	10	0.076
0.8	MAE	-25	66	64	62	67	67	67	67	-11	72	71	70	33	8	0.088
0.8	Med	57	66	64	62	68	67	67	67	59	73	72	71	32	8	0.088
1	MAE	24	65	63	62	68	68	67	67	32	72	71	70	35	6	0.095
1	Med	57	65	63	62	69	68	68	68	60	73	72	71	36	6	0.095
1.2	MAE	-267	64	62	61	68	68	68	67	-323	72	70	70	37	5	0.104
1.2	Med	56	65	62	61	70	69	69	68	59	74	72	71	38	5	0.104
1.4	MAE	-13	62	59	58	69	68	67	67	-12	71	70	69	38	4	0.115
	Med	51	63	60	58	72	71	71	70	53	75	74	73	38	4	0.115
1.6	MAE	5	61	58	56	68	68	67	66	10	71	70	68	40	4	0.124
	Med	49	61	58	56	71	71	70	68	52	74	73	72	40	3	0.124

Table 1: Percent improvement in error over the naive unadjusted difference-in-mean estimator for different estimators: our proposed estimators $\hat{\tau}_{\rm OTC}$, $\hat{\tau}_{\rm SEL}$ and $\hat{\tau}_{\rm DR}$, the estimator proposed in Athey et al. [2020]. Larger percentage decrease means better performance. For reference, the last column shows the error of the naive unadjusted estimator. For our estimators, we fit bridge functions either using no regularization (the "0" column) or ridge regularization with $\lambda = 0.33/n_O^{(a)}, 0.67/n_O^{(a)}$ and $1/n_O^{(a)}$ (the ".33", ".67" and "1" columns) respectively. For Athey et al. [2020], we considered using no regularization ("NR") and using ridge regularization where the regularization parameter is selected by cross validation ("CV").

7.1.2 Results

Table 1 reports the performance of different estimators over 1000 replications of the data subsampling. Each replication results in a different observational dataset \mathcal{D}_O with different number of treated units $n_O^{(1)} < N_1$ and different number of control units $n_O^{(0)} < N_0$. For evaluation we consider two criterions over the 1000 replications: Mean Absolute Error (MAE) and Median of Abolute Errors (MedAE).

In Table 1, we compare the performance of our proposed estimators $\hat{\tau}_{OTC}$, $\hat{\tau}_{SEL}$ and $\hat{\tau}_{DR}$ in Section 5 with two benchmarks: the naive difference-in-mean estimator that uses only the observational dataset and the imputation estimator proposed in Section 4.1 of Athey et al. [2020], which uses both datasets and information of all short-term outcomes $S = (S_1, S_2, S_3)$. The naive estimator completely ignores confounding, and the estimator in Athey et al. [2020] can only account for short-term confounding but not persistent confounding. To evaluate the performance of our estimators and Athey et al. [2020], we consider the percentage decrease in either of our error criteria relative to the naive difference-in-mean estimator. A positive value corresponds to improvement over the naive estimator, and a larger value indicates better performance. A negative value means worse error than the naive estimator.

In our estimators and the imputation estimator in Athey et al. [2020], we need to first estimate some nuisance functions. We specify the outcome bridge function in our estimators and the imputation function in Athey et al. [2020] to be linear functions, and specify the selection bridge function in our estimators to be of the form $q(s_2, s_1, a, x) = \exp(\beta_{2,a}^{\top} s_2 + \beta_{1,a}^{\top} s_1 + \beta_{0,a}^{\top} x + \gamma_a)$. Since these are all simple parametric functions, we do not need the cross-fitting technique described in Section 5, but instead use the same data for nuisance estimation and the final plug-in estimation. To estimate the bridge functions, we employ the generalized method of moment (GMM) approach in Cui et al. [2020]. We consider a standard GMM apporach and the approach with additional ridge regularization, i.e., regularizing the L_2 norms of bridge function coefficients in the GMM objectives, as suggested by Imbens et al. [2021]. When we estimate the bridge function corresponding to the treatment level $a \in \{0, 1\}$, we set the regularization tuning parameter as $\lambda = \lambda_0(n_O^{(a)})^{-1}$ for $\lambda_0 \in \{0, 0.33, 0.67, 1\}$ (note that here $\lambda_0 = 0$ corresponds to no regularization). For the imputation function of Athey et al. [2020], we implement it using either ordinary least squares or cross-validated ridge regression, for which we use the default options in the R package glmnet [Simon et al., 2011].

From Table 1, we observe that with $\lambda_0 \neq 0$, the performance of our proposed estimators $\hat{\tau}_{\text{OTC}}, \hat{\tau}_{\text{SEL}}, \hat{\tau}_{\text{DR}}$ is stable. They consistently outperform the benchmarks, in terms of both criteria. In particular, the doubly robust estimator $\hat{\tau}_{\text{DR}}$ performs the best, reducing the estimation errors of benchmark methods by large margins. Notably, although the benchmark estimator proposed by Athey et al. [2020] improves upon the naive estimator, it is always outperformed by our proposed estimators. This may be due to the fact that the estimator in Athey et al. [2020] cannot handle persistent confounding. We also observe that as the unmeasured confounding becomes stronger (i.e., as η grows), all estimators have higher estimation errors, especially the naive estimator.

We observe that the MAE of our estimators when not using regularization is sometimes worse than the estimator of Athey et al. [2020] and even the naive estimator. This is because unregularized estimators can be unstable and MAE is sensitive to outlier estimates. Indeed, estimating bridge functions requires solving inverse problems defined by conditional moment equations, which can be intrinsically difficult. This problem is common in proximal causal inference, and regularization has been shown to be sometimes key for valid inference [Imbens et al., 2021]. Nevertheless, the MedAE, which is robust to outliers, for our estimators is still lower than the benchmarks. This shows that our proposed estimators, regularized or not, all effectively address the confounding bias. In the supplementary material Appendix F, we heuristically probe the plausibility of Assumptions 6 and 7 according to the characterization of bridge functions in a discrete setting (see example 1). Moreover, we also report the performance of different estimators by varying the number of quarters used for surrogate construction. Our result shows that our approach is consistently more accurate than the approach in Athey et al. [2020].

7.2 A simulation study

In Section 7.1, we focus on parametric estimation of bridge functions. In this part, we use a simulation study to further demonstrate the performance of our approach when bridge functions are nonlinear and estimated by more flexible neural network classes.

Specifically, for both the experimental and observational data, we first generate random vectors \tilde{X} and U from the multivariate normal distribution with mean zero and covariance matrix 0.5**I**, where **I** is an identity matrix with suitable size. We fix the dimension of U as 5 and vary the dimension of \tilde{X} over $\{5, 10, 15, 20\}$. We further generate $Y(a) \in \mathbb{R}, \tilde{S}_1(a) \in \mathbb{R}^5, \tilde{S}_2(a) \in \mathbb{R}, \tilde{S}_3(a) \in \mathbb{R}$

 \mathbb{R}^5 according to the following process:

$$Y(a) = \tau_{y}a + \alpha_{y}^{\top} \tilde{S}_{3}(a) + \beta_{y}^{\top} \tilde{X} + \gamma_{y}^{\top} U + \epsilon_{y},$$

$$\tilde{S}_{j}(a) = \tau_{j}a + \alpha_{j}\tilde{S}_{j-1}(a) + \beta_{j}\tilde{X} + \gamma_{j}U + \epsilon_{j}, \ j \in \{3, 2\}$$

$$\tilde{S}_{1}(a) = \tau_{1}a + \beta_{1}\tilde{X} + \gamma_{1}U + \epsilon_{1},$$

where $\tau_y, (\tau_j, \alpha_y, \beta_y, \gamma_y), (\alpha_j, \beta_j, \gamma_j)$ are scalers, vectors, and matrices of conformable sizes, and ϵ_y, ϵ_j are independent mean-zero Gaussian terms with variance 0.5. We generate the entries in $\tau_y, (\tau_j, \alpha_y, \beta_y, \gamma_y), (\alpha_j, \beta_j, \gamma_j)$ by first drawing numbers from the uniform distribution over the [0, 1] interval and then rescaling them so that the ℓ_2 -norms of the vectors $(\tau_j, \alpha_y, \beta_y, \gamma_y)$ and the columns of $(\alpha_j, \beta_j, \gamma_j)$ are all equal to 0.5. Moreover, we draw the treatment indicator A according to $\mathbb{P}(A=1\mid \tilde{X}, U, G=E)=\frac{1}{2}$ and $\mathbb{P}(A=1\mid \tilde{X}, U, G=O)=(1+\exp(\kappa_1^\top \tilde{X}+\kappa_2^\top U))^{-1}$, where the coefficients κ_1 and κ_2 are similarly generated by sampling and rescaling. According to Example 2 and Appendix B.2, the outcome and selection bridge functions exist under certain rank conditions. Moreover, the outcome bridge function is linear in $\tilde{S}_3, \tilde{S}_2, \tilde{X}, A$ and the selection bridge function is an exponential transformation of a linear function of $\tilde{S}_2, \tilde{S}_1, \tilde{X}, A$. To introduce nonlinear bridge functions, we apply a nonlinear transformation $g(\cdot) = \text{sign}(\cdot) | \cdot|^q$ for $q \in \{1, 1.5, 2\}$ to each element of $\tilde{X}, \tilde{S}_1, \tilde{S}_2, \tilde{S}_3$, leading to X, S_1, S_2, S_3 respectively. This is an invertible transformation that ensures a one-to-one correspondence between the original variables and transformed variables. With these transformations, the bridge functions with respect to the transformed variables X, S_1, S_2, S_3 are linear when q=1 and nonlinear when q=1.5 or 2.

We repeat generating data according to the process above for 200 times. In each replicate, we draw new values for all the model parameters and generate observational data and experimental data accordingly with equal sizes $n_O = n_E = 2000$. We apply our proposed doubly robust estimator and associated confidence intervals to the datasets. We estimate the bridge functions in two ways. One way is to use the minimax learning estimators in Kallus et al. [2021], Dikkala et al. [2020]. Specifically, a minimax bridge function estimator is obtained as the solution to a minimax optimization problem derived from the corresponding conditional moment equation. In our study, we follow Kallus et al. [2021] and specify the outer minimization function class (i.e., the class used to model the bridge function) as a neural network class and the inner maximization function class (i.e., the class used to guarantee the equivalence between the minimax optimization formulation and the conditional moment equation formulation) as a Reproducing Kernel Hilbert Space (RKHS). For implementation details, we refer the readers to the supplementary material. The other way is to use parametric estimators for the bridge functions, where the specifications are identical to those in Section 7.1. These model specifications are correct when q = 1 but wrong when $q \in \{1.5, 2\}$. In both approaches, we also use a ridge regularization with $\lambda = 1/n_O^{(a)}$, i.e., same as the "1" column in Table 1.

Table 2 reports the performance of our estimator and confidence intervals based on two kinds of bridge estimators, over the 200 replicates. When the covariate dimension is relatively low (i.e., $\dim(X) = 5, 10$ or 15), the empirical coverage of the minimax based approach is close to the 95% nominal level in most of the specifications. For the higher dimensional regime $\dim(X) = 20$, the empirical coverage is slightly worse, which could be due to the curse of dimensionality, especially for the inner maximization over RKHS.

When q=1, the average bias of the parametric model based approach is consistently smaller than the minimax based approach. This is expected as the parametric model is correctly specified in this case. In contrast, in the nonlinear settings with q=1.5 and 2, the parametric models are misspecified, so the RMSE and the average bias of the parametric based approach are overall worse

-		$\dim(X) = 5$		$\dim(X)$) = 10	$\dim(X)$) = 15	$\dim(X) = 20$		
\overline{q}		MinMax	Param.	MinMax	Param.	MinMax	Param.	MinMax	Param.	
1	CP	90.0%	94.5%	96.0%	95.5%	94.0%	95.5%	90.5%	95.0%	
	CI Len.	0.541	0.576	0.541	0.578	0.548	0.579	0.546	0.579	
	RMSE	0.157	0.151	0.135	0.139	0.146	0.147	0.155	0.154	
	Bias	0.058	0.017	0.033	0.015	0.036	0.002	0.054	0.001	
1.5	CP	96.5%	97.5%	95.5%	97.0%	95.0%	97.0%	92.5%	97.0%	
	CI Len.	0.619	0.823	0.576	0.828	0.578	0.819	0.574	0.823	
	RMSE	0.159	0.184	0.143	0.199	0.160	0.202	0.157	0.197	
	Bias	0.033	0.031	0.036	0.075	0.034	0.052	0.020	0.056	
2	CP	95.5%	96.0%	94.0%	97.0%	93.0%	97.5%	93.0%	98.0%	
	CI Len.	0.698	2.229	0.611	2.023	0.612	1.842	0.595	2.028	
	RMSE	0.187	0.698	0.157	0.581	0.192	0.521	0.165	0.574	
	Bias	0.073	0.028	0.049	0.085	0.073	0.096	0.032	0.113	

Table 2: The empirical coverage probability (CP) of the 95%-confidence interval and its average length (CI len.), and the root mean squared error (RMSE) and the average absolute bias (Bias) of the doubly robust estimators, with bridge functions estimated by the minimax approach (MinMax) and parametric approach (Param.) respectively. The covariate dimension varies from 5 to 20 and the degree of nonlinearity varies from q = 1 to 2. Here q = 1 corresponds to linear (or exponential linear) bridge functions.

than the minimax based approach, especially for the more nonlinear scenario q=2. This shows the benefit of using flexible function classes to model complex bridge functions.

Interestingly, when q=1.5, 2, the confidence intervals based on the parametric bridge function estimators do not under-cover the truth, even though the corresponding point estimators have larger bias and RMSE. Actually, they tend to over-cover the truth in many specifications. This is perhaps due to the fact that the asymptotic variance tends to be over-estimated, leading to excessively conservative confidence interval lengths. As shown in Table 2, the average confidence interval length produced by the parametric based approach is much larger than the minimax based approach; sometimes, it can even be 4 times larger. Consequently, even with higher average bias, the parametric based approach is still able to get high empirical coverage.

8 Conclusions

In this paper, we consider combining experimental and observational data for long-term causal inference. We are particularly interested in the challenge of persistent confounding, *i.e.*, the presence of unobserved confounders that affect both the short-term and long-term outcomes. To overcome this challenge, we leverage the sequential structure of multiple short-term outcomes and use part of them as proxy variables for the unobserved confounders. We propose three novel identification strategies for the average long-term treatment effect. Based on each of them, we design flexible treatment effect estimators and inference methods, for which we provide asymptotic guarantees. Our results show that the long-term treatment effect can be identified and estimated under much more general conditions than before.

Beyond these specific results, our work more broadly reveals an interesting role for the structure of short-term outcomes in long-term causal inference. To the best of our knowledge, the structure of repeated outcome measurements is largely unexplored in the long-term causal inference literature.

We hope that our work will inspire other researchers to study other plausible structures for shortterm outcomes and benefits these can have for long-term causal inference.

References

- Chunrong Ai and Xiaohong Chen. Efficient estimation of models with conditional moment restrictions containing unknown functions. *Econometrica*, 71(6):1795–1843, 2003.
- Joshua D Angrist and Alan B Krueger. The effect of age at school entry on educational attainment: an application of instrumental variables with moments from two samples. *Journal of the American statistical Association*, 87(418):328–336, 1992.
- Susan Athey, Raj Chetty, Guido W Imbens, and Hyunseung Kang. The surrogate index: Combining short-term proxies to estimate long-term treatment effects more rapidly and precisely. *NBER Working Paper*, (w26463), 2019.
- Susan Athey, Raj Chetty, and Guido Imbens. Combining experimental and observational data to estimate treatment effects on long term outcomes, 2020.
- Keith Battocchi, Eleanor Dillon, Maggie Hei, Greg Lewis, Miruna Oprescu, and Vasilis Syrgkanis. Estimating the long-term effects of novel treatments. *Advances in Neural Information Processing Systems*, 34, 2021.
- Andrew Bennett and Nathan Kallus. The variational method of moments. arXiv preprint arXiv:2012.09422, 2020.
- Andrew Bennett and Nathan Kallus. Proximal reinforcement learning: Efficient off-policy evaluation in partially observed markov decision processes. arXiv preprint arXiv:2110.15332, 2021.
- Andrew Bennett, Nathan Kallus, and Tobias Schnabel. Deep generalized method of moments for instrumental variable analysis. In *Advances in Neural Information Processing Systems 32*, pages 3564–3574. 2019.
- Andrew Bennett, Nathan Kallus, Xiaojie Mao, Whitney Newey, Vasilis Syrgkanis, and Masatoshi Uehara. Inference on strongly identified functionals of weakly identified functions. arXiv e-prints, pages arXiv-2208, 2022.
- Richard Blundell and James L Powell. Endogeneity in nonparametric and semiparametric regression models. 2003.
- Hengrui Cai, Wenbin Lu, and Rui Song. Coda: Calibrated optimal decision making with multiple data sources and limited outcome. arXiv preprint arXiv:2104.10554, 2021a.
- Hengrui Cai, Rui Song, and Wenbin Lu. Gear: On optimal decision making with auxiliary data. Stat, 10(1):e399, 2021b.
- Marine Carrasco, jean-pierre Florens, and Eric Renault. Chapter 77 linear inverse problems in structural econometrics estimation based on spectral decomposition and regularization. *Handbook of Econometrics*, 6:5633–5751, 12 2007.
- Hua Chen, Zhi Geng, and Jinzhu Jia. Criteria for surrogate end points. Journal of the Royal Statistical Society: Series B (Statistical Methodology), 69(5):919–932, 2007.

- Jiafeng Chen and David M Ritzwoller. Semiparametric estimation of long-term treatment effects. arXiv preprint arXiv:2107.14405, 2021.
- Shuxiao Chen, Bo Zhang, and Ting Ye. Minimax rates and adaptivity in combining experimental and observational data. arXiv preprint arXiv:2109.10522, 2021.
- Xiaohong Chen and Demian Pouzo. Estimation of nonparametric conditional moment models with possibly nonsmooth generalized residuals. *Econometrica*, 80(1):277–321, 2012.
- David Cheng and Tianxi Cai. Adaptive combination of randomized and observational data. arXiv preprint arXiv:2111.15012, 2021.
- V Chernozhukov, W Newey, J Robins, and R Singh. Double/de-biased machine learning of global and local parameters using regularized riesz representers. *stat*, 1050:9, 2019.
- Victor Chernozhukov, Iván Fernández-Val, Whitney Newey, Sami Stouli, and Francis Vella. Semi-parametric estimation of structural functions in nonseparable triangular models. *Quantitative Economics*, 11(2):503–533, 2020.
- Raj Chetty, John N Friedman, Nathaniel Hilger, Emmanuel Saez, Diane Whitmore Schanzenbach, and Danny Yagan. How does your kindergarten classroom affect your earnings? evidence from project star. *The Quarterly journal of economics*, 126(4):1593–1660, 2011.
- Bénédicte Colnet, Imke Mayer, Guanhua Chen, Awa Dieng, Ruohong Li, Gaël Varoquaux, Jean-Philippe Vert, Julie Josse, and Shu Yang. Causal inference methods for combining randomized trials and observational studies: a review. arXiv preprint arXiv:2011.08047, 2020.
- Yifan Cui, Hongming Pu, Xu Shi, Wang Miao, and Eric Tchetgen Tchetgen. Semiparametric proximal causal inference. arXiv preprint arXiv:2011.08411, 2020.
- Serge Darolles, Yanqin Fan, Jean-Pierre Florens, and Eric Renault. Nonparametric instrumental regression. *Econometrica*, 79(5):1541–1565, 2010.
- Ben Deaner. Proxy controls and panel data. arXiv preprint arXiv:1810.00283, 2021.
- Nishanth Dikkala, Greg Lewis, Lester Mackey, and Vasilis Syrgkanis. Minimax estimation of conditional moment models. In *Advances in Neural Information Processing Systems*, volume 33, pages 12248–12262, 2020.
- Oliver Dukes, Ilya Shpitser, and Eric J Tchetgen Tchetgen. Proximal mediation analysis. arXiv preprint arXiv:2109.11904, 2021.
- Jean-Pierre Florens, James J Heckman, Costas Meghir, and Edward Vytlacil. Identification of treatment effects using control functions in models with continuous, endogenous treatment and heterogeneous effects. *Econometrica*, 76(5):1191–1206, 2008.
- Constantine E Frangakis and Donald B Rubin. Principal stratification in causal inference. *Biometrics*, 58(1):21–29, 2002.
- AmirEmad Ghassami, Ilya Shpitser, and Eric Tchetgen Tchetgen. Proximal causal inference with hidden mediators: Front-door and related mediation problems. arXiv preprint arXiv:2111.02927, 2021a.

- AmirEmad Ghassami, Andrew Ying, Ilya Shpitser, and Eric Tchetgen Tchetgen. Minimax kernel machine learning for a class of doubly robust functionals. 2021b.
- AmirEmad Ghassami, Alan Yang, David Richardson, Ilya Shpitser, and Eric Tchetgen Tchetgen. Combining experimental and observational data for identification and estimation of long-term causal effects. 2022.
- Bryan S Graham, Cristine Campos de Xavier Pinto, and Daniel Egel. Efficient estimation of data combination models by the method of auxiliary-to-study tilting (ast). *Journal of Business & Economic Statistics*, 34(2):288–301, 2016.
- Somit Gupta, Ronny Kohavi, Diane Tang, Ya Xu, Reid Andersen, Eytan Bakshy, Niall Cardin, Sumita Chandran, Nanyu Chen, Dominic Coey, et al. Top challenges from the first practical online controlled experiments summit. *ACM SIGKDD Explorations Newsletter*, 21(1):20–35, 2019.
- Peter Hall and Joel L. Horowitz. Nonparametric methods for inference in the presence of instrumental variables. The Annals of Statistics, 33(6):2904 2929, 2005. doi: 10.1214/009053605000000714. URL https://doi.org/10.1214/009053605000000714.
- Lars Peter Hansen. Large sample properties of generalized method of moments estimators. *Econometrica: Journal of the Econometric Society*, pages 1029–1054, 1982.
- Jason Hartford, Greg Lewis, Kevin Leyton-Brown, and Matt Taddy. Deep IV: A flexible approach for counterfactual prediction. In *Proceedings of the 34th International Conference on Machine Learning*, volume 70, pages 1414–1423, 2017.
- Henning Hohnhold, Deirdre O'Brien, and Diane Tang. Focusing on the long-term: It's good for users and business. In *Proceedings of the 21th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, pages 1849–1858, 2015.
- V Joseph Hotz, Guido W Imbens, and Jacob A Klerman. Evaluating the differential effects of alternative welfare-to-work training components: A reanalysis of the california gain program. *Journal of Labor Economics*, 24(3):521–566, 2006.
- Guido Imbens and Susan Athey. Identification and inference in nonlinear difference-in-difference models. *Econometrica*, 74:431–497, 02 2006. doi: 10.2139/ssrn.311920.
- Guido Imbens, Nathan Kallus, and Xiaojie Mao. Controlling for unmeasured confounding in panel data using minimal bridge functions: From two-way fixed effects to factor models. arXiv preprint arXiv:2108.03849, 2021.
- Guido W Imbens and Whitney K Newey. Identification and estimation of triangular simultaneous equations models without additivity. *Econometrica*, 77(5):1481–1512, 2009.
- Marshall M Joffe and Tom Greene. Related causal frameworks for surrogate outcomes. *Biometrics*, 65(2):530–538, 2009.
- Nathan Kallus and Xiaojie Mao. On the role of surrogates in the efficient estimation of treatment effects with limited outcome data. arXiv preprint arXiv:2003.12408, 2020.
- Nathan Kallus, Aahlad Manas Puli, and Uri Shalit. Removing hidden confounding by experimental grounding. Advances in neural information processing systems, 31, 2018.

- Nathan Kallus, Xiaojie Mao, and Masatoshi Uehara. Causal inference under unmeasured confounding with negative controls: A minimax learning approach, 2021.
- Richard Kay. A markov model for analysing cancer markers and disease states in survival studies. *Biometrics*, pages 855–865, 1986.
- Ron Kohavi, Alex Deng, Brian Frasca, Roger Longbotham, Toby Walker, and Ya Xu. Trustworthy online controlled experiments: Five puzzling outcomes explained. In *Proceedings of the 18th ACM SIGKDD international conference on Knowledge discovery and data mining*, pages 786–794, 2012.
- Rainer Kress, V Maz'ya, and V Kozlov. Linear integral equations, volume 82. Springer, 1989.
- Yiyuan Liu, Minghui Wang, Andrew D Morris, Alex SF Doney, Graham P Leese, Ewan R Pearson, and Colin NA Palmer. Glycemic exposure and blood pressure influencing progression and remission of diabetic retinopathy: a longitudinal cohort study in godarts. *Diabetes Care*, 36(12): 3979–3984, 2013.
- Guillermo Marshall and Richard H Jones. Multi-state models and diabetic retinopathy. *Statistics in medicine*, 14(18):1975–1983, 1995.
- Afsaneh Mastouri, Yuchen Zhu, Limor Gultchin, Anna Korba, Ricardo Silva, Matt Kusner, Arthur Gretton, and Krikamol Muandet. Proximal causal learning with kernels: Two-stage estimation and moment restriction. In *International Conference on Machine Learning*, pages 7512–7523. PMLR, 2021.
- Wang Miao and Eric Tchetgen Tchetgen. A confounding bridge approach for double negative control inference on causal effects (supplement and sample codes are included). arXiv preprint arXiv:1808.04945, 2018.
- Wang Miao, Zhi Geng, and Eric Tchetgen. Identifying causal effects with proxy variables of an unmeasured confounder. *Biometrika*, 105, 09 2016. doi: 10.1093/biomet/asy038.
- Sandeep Mohapatra, Scott Rozelle, and Rachael Goodhue. The rise of self-employment in rural china: development or distress? *World Development*, 35(1):163–181, 2007.
- Kenichi Nagasawa. Treatment effect estimation with noisy conditioning variables. arXiv preprint arXiv:1811.00667, 2018.
- Whitney Newey and Sami Stouli. Control variables, discrete instruments, and identification of structural functions. *Journal of Econometrics*, 222(1):73–88, 2021.
- Whitney K. Newey and James L. Powell. Instrumental variable estimation of nonparametric models. *Econometrica*, 71(5):1565–1578, 2003.
- James M Poterba and Lawrence H Summers. Reporting errors and labor market dynamics. *Econometrica: Journal of the Econometric Society*, pages 1319–1338, 1986.
- Ross L Prentice. Surrogate endpoints in clinical trials: definition and operational criteria. *Statistics in medicine*, 8(4):431–440, 1989.
- Brenda L Price, Peter B Gilbert, and Mark J van der Laan. Estimation of the optimal surrogate based on a randomized trial. *Biometrics*, 74(4):1271–1281, 2018.

- Zhengling Qi, Rui Miao, and Xiaoke Zhang. Proximal learning for individualized treatment regimes under unmeasured confounding. arXiv preprint arXiv:2105.01187, 2021.
- Thomas S Richardson and James M Robins. Single world intervention graphs (swigs): A unification of the counterfactual and graphical approaches to causality. Center for the Statistics and the Social Sciences, University of Washington Series. Working Paper, 128(30):2013, 2013.
- Evan Rosenman, Guillaume Basse, Art Owen, and Michael Baiocchi. Combining observational and experimental datasets using shrinkage estimators. arXiv preprint arXiv:2002.06708, 2020.
- Evan TR Rosenman, Art B Owen, Mike Baiocchi, and Hailey R Banack. Propensity score methods for merging observational and experimental datasets. *Statistics in Medicine*, 41(1):65–86, 2022.
- Donald B Rubin. Estimating causal effects of treatments in randomized and nonrandomized studies. Journal of educational Psychology, 66(5):688, 1974.
- Xu Shi, Wang Miao, Jennifer C. Nelson, and Eric J. Tchetgen Tchetgen. Multiply robust causal inference with double-negative control adjustment for categorical unmeasured confounding. *Journal of The Royal Statistical Society Series B-statistical Methodology*, 82(2):521–540, 2020.
- Xu Shi, Wang Miao, Mengtong Hu, and Eric Tchetgen Tchetgen. Theory for identification and inference with synthetic controls: A proximal causal inference framework. arXiv preprint arXiv:2108.13935, 2021.
- Noah Simon, Jerome Friedman, Trevor Hastie, and Rob Tibshirani. Regularization paths for cox's proportional hazards model via coordinate descent. *Journal of Statistical Software*, 39(5):1–13, 2011. URL https://www.jstatsoft.org/v39/i05/.
- Rahul Singh. Kernel methods for unobserved confounding: Negative controls, proxies, and instruments. arXiv preprint arXiv:2012.10315, 2020.
- Rahul Singh. A finite sample theorem for longitudinal causal inference with machine learning: Long term, dynamic, and mediated effects. arXiv preprint arXiv:2112.14249, 2021.
- Rahul Singh. Generalized kernel ridge regression for long term causal inference: Treatment effects, dose responses, and counterfactual distributions. arXiv preprint arXiv:2201.05139, 2022.
- Rahul Singh, Maneesh Sahani, and Arthur Gretton. Kernel instrumental variable regression. In *Advances in Neural Information Processing Systems*, volume 32, 2019.
- Peter Spirtes, Clark N Glymour, Richard Scheines, and David Heckerman. Causation, prediction, and search. MIT press, 2000.
- Eric J Tchetgen Tchetgen, Andrew Ying, Yifan Cui, Xu Shi, and Wang Miao. An introduction to proximal causal learning. arXiv e-prints, pages arXiv-2009, 2020.
- Guy Tennenholtz, Uri Shalit, and Shie Mannor. Off-policy evaluation in partially observable environments. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 34, pages 10276–10283, 2020.
- Anastasios Tsiatis. Semiparametric theory and missing data. Springer Science & Business Media, 2007.
- Aad W Van der Vaart. Asymptotic statistics, volume 3. Cambridge university press, 2000.

- Tyler J VanderWeele. Surrogate measures and consistent surrogates. *Biometrics*, 69(3):561–565, 2013.
- Xuan Wang, Layla Parast, Lu Tian, and Tianxi Cai. Model-free approach to quantifying the proportion of treatment effect explained by a surrogate marker. *Biometrika*, 107(1):107–122, 2020.
- Christopher J Weir and Rosalind J Walley. Statistical evaluation of biomarkers as surrogate endpoints: a literature review. *Statistics in medicine*, 25(2):183–203, 2006.
- Jeffrey M Wooldridge. Control function methods in applied econometrics. *Journal of Human Resources*, 50(2):420–445, 2015.
- Liyuan Xu, Heishiro Kanagawa, and Arthur Gretton. Deep proxy causal learning and its application to confounded bandit policy evaluation. *Advances in Neural Information Processing Systems*, 34, 2021.
- Jeremy Yang, Dean Eckles, Paramveer Dhillon, and Sinan Aral. Targeting for long-term outcomes. arXiv preprint arXiv:2010.15835, 2020a.
- Shu Yang and Peng Ding. Combining multiple observational data sources to estimate causal effects. Journal of the American Statistical Association, 2019.
- Shu Yang, Donglin Zeng, and Xiaofei Wang. Elastic integrative analysis of randomized trial and real-world data for treatment heterogeneity estimation. arXiv preprint arXiv:2005.10579, 2020b.
- Shu Yang, Donglin Zeng, and Xiaofei Wang. Improved inference for heterogeneous treatment effects using real-world data subject to hidden confounding. arXiv preprint arXiv:2007.12922, 2020c.
- Andrew Ying, Wang Miao, Xu Shi, and Eric J Tchetgen Tchetgen. Proximal causal inference for complex longitudinal studies. arXiv preprint arXiv:2109.07030, 2021.
- Wenjing Zheng and Mark J Laan. Cross-validated targeted minimum-loss-based estimation. In *Targeted Learning*, pages 459–474. Springer, 2011.

A Comparison to Other Identifying Conditions

To identify the average long-term treatment effect using data combination, restrictions must be imposed on unobserved confounders. In this paper, we crucially leverage an assumed sequential structure in the short-term outcomes and an assumption that these are sufficiently strong proxies (our Assumptions 4 and 5). In this section, we compare to two other sets of assumptions that, in addition to Assumptions 1 to 3, minimally ensure identification, and we discuss their relationship to persistent confounding. Each of the following provide an alternative setting that is *just identified*, meaning dropping any one assumption breaks identification. Indeed, in our paper we needed Assumptions 4 and 5 for identification.

A.1 Comparison to Athey et al. [2020]

Athey et al. [2020] assume latent unconfoundedness: $Y(a) \perp A \mid S(a), X, G = O$. The assumption, which makes no explicit reference to presence or absence of persistent confounding, states that, were it observed, controlling for (S(a), X) would be sufficient. Along with Assumptions 1 to 3 (or, Assumption 9), they show this assumption ensures identification.

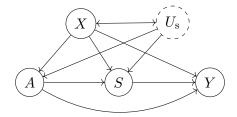
There are many ways to potentially satisfy this abstract assumption. One possibility is if $S(a) = f_a(U)$ is an invertible transformation of U. Such a production-function approach calls to mind, for example, assumption 3.2 of Imbens and Athey [2006]. This, however, precludes lossyness or noise in the relationship between short-term outcomes and confounders. Alternatively, we can consider restrictions encoded solely by causal diagrams that would ensure latent unconfoundedness holds. One such diagram is shown in Figure 4a: here the unobserved confounders are only short-term confounders (U_s) in they that can only affect the treatment and short-term outcomes, but not the long-term outcome. Another diagram is shown in Figure 4b: here the unobserved confounders are only outcome confounders (U_o) in that they simultaneously affect the short-term and long-term outcomes, but not the treatment.

Latent unconfoundedness generally may not hold in a diagram where confounders are persistent (Figure 1), and in fact it does not whenever a distribution is "well-represented" by such a diagram. In Figure 5a, we duplicate the single world intervention graph in Figure 2a for the observational data. This summarizes the statistical independences among the potential outcomes and other variables in the observational data. In this graph, the path $A \leftarrow U \rightarrow Y(a)$ is not blocked by the nodes S(a) and X, so A and Y(a) are not d-separated by S(a) and X. This means that the latent unconfoundedness assumption in Athey et al. [2020] is violated when the distribution of the random variables (X, U, A, S(a), Y(a)) is faithful to the single world intervention graph in Figure 5a [Spirtes et al., 2000], roughly meaning that the graph is minimal for the distribution.²

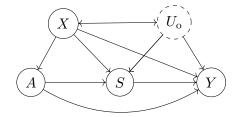
A.2 Comparison to Athey et al. [2019]

Athey et al. [2019] assume that the long-term outcome is independent of the treatment given the short-term outcomes, $A \perp Y \mid S, X, G = E$. This is based on the surrogate criterion proposed by Prentice [1989]. Crucially they show this condition enables identification even when A is missing in the observational data, which can be extremely practical. This condition, however, rules out any direct effect of the treatment on the long-term outcome and any confounding between short-term and long-term outcomes, as might be induced by a persistent confounder.

Formally, we say that a distribution \mathbb{P} on the nodes of graph \mathcal{G} is faithful to the graph \mathcal{G} when for any random variables (A, B, C) in the graph, $A \perp B \mid C$ under the distribution \mathbb{P} if and only if A and B are d-separated by C in the graph \mathcal{G} [Spirtes et al., 2000].

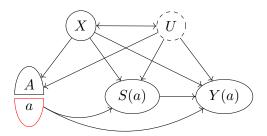


(a) Short-term confounders U_s .

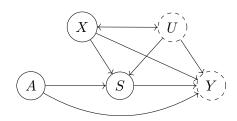


(b) Outcome confounders U_0 .

Figure 4: Short-term confounders and outcome confounders in the observational data.



(a) Single world intervention graph for observational data with persistent confounders.



(b) Causal graph for experimental data with persistent confounders.

Figure 5: Graphs for persistent confounders.

In Figure 5b, we duplicate the causal diagram in Figure 1b, which describes the causal relationship between variables in the experimental data. In the setting of Figure 5b, the surragacy condition is violated when the distribution of the random variables (X, U, A, S, Y) is faithful. (Note we do not use a single world intervention graph here as the assumption is made on factual variables, rather than on potential outcomes.) Indeed, in Figure 5b, the short-term outcomes S are colliders between the treatment A and the persistent confounders U, so conditioning on S induces dependence between the treatment A and the long-term outcome Y. Moreover, the treatment A can also have direct causal effect on the long-term outcome Y. Therefore, unless the dependence due to the direct causal effect of the treatment and the dependence due to conditioning on colliders happen to cancel with each other (which cannot happen if the distribution is faithful), the surrogacy condition in Athey et al. [2019] is violated.

B Selection Bridge Functions in Special Examples

B.1 Discrete Setting

Recall that in Example 1, we consider $S_1 = S_2 = S_3 = \{s_{(j)} : j = 1, \dots, M_s\}$ and $\mathcal{U} = \{u_{(k)} : k = 1, \dots, M_u\}$. For any $s_2 \in S_2$, $a \in \mathcal{A}$, $x \in \mathcal{X}$, let $P(\mathbf{S}_1 \mid s_2, a, \mathbf{U}, x) \in \mathbb{R}^{M_s \times M_u}$ denote the matrix whose (j, k)th element is

$$\mathbb{P}\left(S_{1} = s_{(j)} \mid S_{2} = s_{2}, A = a, U = u_{(k)}, X = x, G = O\right),\,$$

and $r(s_2, \mathbf{U}, x; a) \in \mathbb{R}^{M_u}$ denote the vector whose kth element is

$$p(s_2, u_{(k)}, x \mid a, G = E)/p(s_2, u_{(k)}, x \mid a, G = O).$$

The existence of a selection bridge function is equivalent to the existence of a solution $z \in \mathbb{R}^{M_s}$ to the following linear equation system for any $s_2 \in \mathcal{S}_2, a \in \mathcal{A}, x \in \mathcal{X}$:

$$[P(\mathbf{S}_1 \mid s_2, a, \mathbf{U}, x)]^{\top} z = r(s_2, \mathbf{U}, x; a).$$

One sufficient condition for the existence of solutions to the equation above is that the matrix $P(\mathbf{S}_1 \mid s_2, a, \mathbf{U}, x)$ has full column rank for any $s_2 \in \mathcal{S}_2, a \in \mathcal{A}, x \in \mathcal{X}$. This full column rank condition means that S_1 is strongly informative for U.

B.2 Linear Models

Recall that in Example 2, (Y, S_3, S_2, S_1) are generated from the following linear structural equation system:

$$Y = \tau_y A + \alpha_y^{\top} S_3 + \beta_y^{\top} X + \gamma_y^{\top} U + \epsilon_y,$$

$$S_j = \tau_j A + \alpha_j S_{j-1} + \beta_j X + \gamma_j U + \epsilon_j, \ j \in \{3, 2\}$$

$$S_1 = \tau_1 A + \beta_1 X + \gamma_1 U + \epsilon_1,$$

where τ_y , $(\tau_j, \alpha_y, \beta_y, \gamma_y)$, $(\alpha_j, \beta_j, \gamma_j)$ are scalers, vectors, and matrices of conformable sizes respectively, and ϵ_y , ϵ_j are independent mean-zero noise terms such that $\epsilon_y \perp (S, A, U, X)$ and $\epsilon_j \perp (S_{j-1}, \ldots, S_1, A, U, X)$.

We further assume $\mathbb{P}(A = 1 \mid U, X, G = E) = 1/2$ and $\mathbb{P}(A = 1 \mid U, X, G = O) = (1 + \exp(\kappa_1^\top U + \kappa_2^\top X))^{-1}$. We also assume that $(\epsilon_3, \epsilon_2, \epsilon_1)$ follows a joint Gaussian distribution with zero mean and a diagonal covariance matrix. Denote the covariance matrix for ϵ_j as $\sigma_j^2 I_j$ for $j = 1, \ldots, 3$ where I_j is an identity matrix of formable size.

Proposition 1. Given the data generating process described above, $S_1 \mid S_2, A, U, X, G = O$ follows a Gaussian distribution with conditional expectation

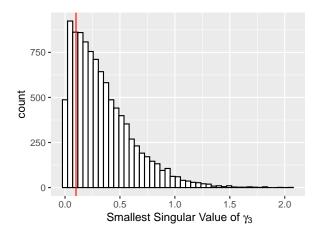
$$\mathbb{E}[S_1 \mid S_2, A, U, X, G = O] = \lambda_1 S_2 + \lambda_2 A + \lambda_3 X + \lambda_4 U$$

where

$$\begin{split} \lambda_1 &= \sigma_1^2 \alpha_2^\top \left(\sigma_1^2 \alpha_2 \alpha_2^\top + \sigma_2^2 I_2 \right)^{-1}, \\ \lambda_2 &= \left(I_1 - \sigma_1^2 \alpha_2^\top \left(\sigma_1^2 \alpha_2 \alpha_2^\top + \sigma_2^2 I_2 \right)^{-1} \alpha_2 \right) \tau_1 - \sigma_1^2 \alpha_2^\top \left(\sigma_1^2 \alpha_2 \alpha_2^\top + \sigma_2^2 I_2 \right)^{-1} \tau_2 \\ \lambda_3 &= \left(I_1 - \sigma_1^2 \alpha_2^\top \left(\sigma_1^2 \alpha_2 \alpha_2^\top + \sigma_2^2 I_2 \right)^{-1} \alpha_2 \right) \beta_1 - \sigma_1^2 \alpha_2^\top \left(\sigma_1^2 \alpha_2 \alpha_2^\top + \sigma_2^2 I_2 \right)^{-1} \beta_2 \\ \lambda_4 &= \left(I_1 - \sigma_1^2 \alpha_2^\top \left(\sigma_1^2 \alpha_2 \alpha_2^\top + \sigma_2^2 I_2 \right)^{-1} \alpha_2 \right) \gamma_1 - \sigma_1^2 \alpha_2^\top \left(\sigma_1^2 \alpha_2 \alpha_2^\top + \sigma_2^2 I_2 \right)^{-1} \gamma_2. \end{split}$$

When λ_4 has full column rank, then for any $\tilde{\theta}_1$ such that $\tilde{\theta}_1^{\top} \lambda_4 = \kappa_2^{\top}$ and $a \in \mathcal{A}$, there exists a selection bridge function of the following form for some matrices $\tilde{\theta}_2, \tilde{\theta}_0$ of conformable sizes and some constants $c_{1,a}, c_{0,a}$:

$$q_0(S_2, S_1, a, X) = c_{1,a} \exp\left((-1)^a \left(\tilde{\theta}_2^\top S_2 + \tilde{\theta}_1^\top S_1 + \tilde{\theta}_0^\top X\right)\right) + c_{0,a}.$$



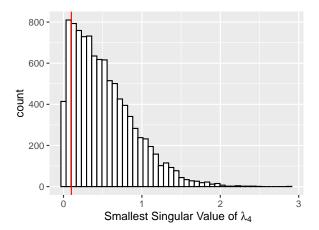


Figure 6: Distributions of the smallest singular values of the λ_3 and γ_4 matrices over 10000 replications. The vertical bar corresponds to the 0.1 singular value.

According to Proposition 1 and example 2, the outcome bridge function and selection bridge function exist in this linear model setting if the matrix γ_3 and λ_4 have full column rank. To illustrate these existence conditions, we also run a simple simulation study. Specifically, we generate data according to the linear model above. We set $\dim(S_1) = \dim(S_2) = \dim(S_3) = 2$, draw all coefficients $\tau, \alpha, \beta, \gamma$'s from the standard normal distribution, and all noise terms from the mean-zero normal distribution with variance 0.5. We generate 10000 instances, compute the smallest singular values of the corresponding γ_3 and λ_4 matrices, and report their distributions in Figure 6. We observe that the smallest singular value of γ_3 is larger than 0.1 around 88% of time and the smallest singular value of λ_4 is larger than 0.1 around 81% of time. These show that the existence of bridge functions in Assumptions 6 and 7 may not always hold but it does hold in quite many scenarios.

C Completeness Conditions and Existence of Bridge Functions

The conditional moment equations in Equations (7) and (11) that define outcome bridge functions and selection bridge functions are Fredholm integral equations of the first kind. Following Miao et al. [2016], we characterize the existence of their solutions (*i.e.*, the outcome and selection bridge functions) by singular value decomposition of compact operators [Carrasco et al., 2007].

Let $L_2(p(z))$ denote the space of all square integrable functions of z with respect to the distribution p(z). It is a Hilbert space with inner product $\langle f_1, f_2 \rangle = \int f_1(z) f_2(z) p(z) \, dz$. Consider linear operators $\mathcal{T}_{s_2,a,x} : L_2(p(s_3 \mid s_2, a, x)) \to L_2(p(u \mid s_2, a, x))$, $\mathcal{T}'_{s_2,a,x} : L_2(p(s_1 \mid s_2, a, x)) \to L_2(p(u \mid s_2, a, x))$ defined as follows:

$$[\mathcal{T}_{s_2,a,x}h] (s_2,a,u,x) = \mathbb{E} [h(S_3,S_2,A,X) \mid S_2 = s_2, A = a, U = u, X = x, G = O],$$

$$[\mathcal{T}'_{s_2,a,x}q] (s_2,a,u,x) = \mathbb{E} [q(S_2,S_1,A,X) \mid S_2 = s_2, A = a, U = u, X = x, G = O].$$

Assumption 11. For any $s_2 \in \mathcal{S}_2$, $a \in \mathcal{A}$, $x \in \mathcal{A}$,

- 1. $\iint p(s_3 \mid s_2, a, u, x) p(u \mid s_3, s_2, a, x) ds_3 du < \infty$.
- 2. $\iint p(s_1 \mid s_2, a, u, x) p(u \mid s_1, s_2, a, x) ds_1 du < \infty$.

According to Example 2.3 in Carrasco et al. [2007], Assumption 11 ensures that for any $s_2 \in \mathcal{S}_2, a \in \mathcal{A}, x \in \mathcal{A}, \mathcal{T}_{s_2,a,x}, \mathcal{T}'_{s_2,a,x}$ are both compact operators. Then by Theorem 2.41 in

Carrasco et al. [2007], both of them admit singular value decomposition. Namely, there exist $(\lambda_{s_2,a,x,j}, \psi_{s_2,a,x,j}, \phi_{s_2,a,x,j}, \phi_{s_2,a,x,j}, \phi_{s_2,a,x,j}, \phi_{s_2,a,x,j})_{j=1}^{\infty}$ such that for any j,

$$\mathcal{T}_{s_2,a,x}\psi_{s_2,a,x,j} = \lambda_{s_2,a,x,j}\phi_{s_2,a,x,j} \mathcal{T}'_{s_2,a,x}\psi'_{s_2,a,x,j} = \lambda'_{s_2,a,x,j}\phi'_{s_2,a,x,j}.$$

Assumption 12. For any $s_2 \in \mathcal{S}_2$, $a \in \mathcal{A}$, $x \in \mathcal{A}$,

1.
$$\mathbb{E}[Y \mid s_2, a, u, x, G = O]$$
 and $\frac{p(s_2, u, x \mid a, G = E)}{p(s_2, u, x \mid a, G = O)}$ both belong to $L_2(p(u \mid s_2, a, x))$.

2.
$$\sum_{j=1}^{n} \lambda_{s_2,a,x,j}^{-2} |\langle \mathbb{E}[Y | s_2, a, u, x, G = O], \phi_{s_2,a,x,j} \rangle|^2 < \infty.$$

3.
$$\sum_{j=1}^{n} \lambda'^{-2}_{s_2,a,x,j} \left| \left\langle \frac{p(s_2,u,x|a,G=E)}{p(s_2,u,x|a,G=O)}, \phi'_{s_2,a,x,j} \right\rangle \right|^2 < \infty.$$

Under regularity conditions in Assumptions 11 and 12, it can be shown that completeness conditions in Assumption 5 guarantee the existence of bridge functions.

Proposition 2 (Existence of Bridge Functions). Suppose that Assumptions 11 and 12 hold.

- 1. If the completeness condition in Assumption 5 condition 1 holds, then there exists an outcome bridge function h₀ satisfying Equation (7).
- 2. If the completeness condition in Assumption 5 condition 2 holds, then there exists an outcome bridge function q₀ satisfying Equation (11).

Proposition 2 can be proved by Picard's Theorem [Kress et al., 1989, Theorem 15.18]. See Lemma 2 in Miao et al. [2016] or Lemma 13 and 14 in Kallus et al. [2021] for details.

D Relaxing Assumptions 2 and 3

In this section, we present additional identification results under Assumptions 9 and 10 instead of the stronger conditions in Assumptions 2 and 3, and discuss their relations to the existing literature. We also discuss how to estimate the average long-term treatment effect in this case, based on the doubly robust identification strategy in Theorem 8.

D.1 Identification

In Theorem 8, we consider extending the doubly robust identification strategy in Theorem 3, which involves both outcome and selection bridge functions. We now show that based on Theorem 8, we can also extend Theorems 1 and 2.

Corollary 1. Suppose Assumptions 1, 4, 9 and 10 hold.

1. If further the completeness condition in Assumption 5 condition 2 and Assumption 6 hold, then the average long-term treatment effect can be identified by any function h₀ that satisfies Equation (9):

$$\tau = \sum_{a \in \{0,1\}} (-1)^{1-a} \mathbb{E} \left[\mathbb{E} \left[h_0(S_3, S_2, A, X) \mid A = a, X, G = E \right] \mid G = O \right].$$
 (20)

2. If further the completeness condition in Assumption 5 condition 1 and Assumption 7 hold, then the average long-term treatment effect can be identified by any function q₀ that satisfies Equation (12) or Equation (13):

$$\tau = \sum_{a \in \{0,1\}} (-1)^{1-a} \mathbb{E} \left[\frac{\mathbb{P}(G = E \mid A = a) \mathbb{P}(G = O \mid X)}{\mathbb{P}(G = O \mid A = a) \mathbb{P}(G = E \mid X)} \frac{\mathbb{I}[A = a]}{\mathbb{P}(A = a \mid X, G = E)} \times q_0(S_2, S_1, A, X) Y \mid G = O \right]$$
(21)

Proof for Corollary 1. Obviously, Equation (20) can be proved by setting $h = h_0, q = 0$ in Theorem 8 and Equation (21) can be proved by setting $q = q_0, h = 0$ in Theorem 8.

We note that the two identification strategies in Corollary 1 are closely related to those in Athey et al. [2020], Ghassami et al. [2022]. As we discussed in Remark 1, when there is no persistent confounder, we can let $S_1 = S_3 = \emptyset$ and $S = S_2$. Then $h_0(S_2, A, X) = \mathbb{E}[Y \mid S, A, X, G = O]$ is the unique solution to Equation (9). As a result, the identification strategy in Equation (20) exactly recovers the identification strategy in Theorem 1 in Athey et al. [2020]. Moreover, in the setup in Ghassami et al. [2022], if we use S_3 as their short-term outcomes, S_1 as their auxiliary proxies, and condition on S_2 appropriately, then the identification strategies in their Theorems 9 and 10 coincide with ours in Equations (20) and (21) respectively. Compared to Ghassami et al. [2022], we only require short-term outcomes without needing to search for additional external proxies. See also discussions in Section 2.2 for additional comparisons.

Under the weaker conditions in Assumptions 9 and 10, Corollary 1 and theorem 8 shows that we need more complex identification strategies for the average long-term treatment effect over the observational data distribution. Actually, even in this case, the simpler identification strategies in Sections 4.1 to 4.3 are still useful. Below we show that under an additional assumption, they can identify average long-term treatment effect over the experimental data distribution.

Corollary 2. Suppose Assumptions 1, 4 to 7, 9 and 10 hold and $Y(a) \perp G \mid S(a), U, X$. Then Equation (10) in Theorem 1, Equation (14) in Theorem 2 and Equation (15) in Theorem 3 all identify the average long-term treatment effect over the experimental data distribution, i.e.,

$$\tau_E = \mathbb{E}\left[Y(1) - Y(0) \mid G = E\right],\,$$

In Corollary 2, we still assume the weaker conditions in Assumptions 9 and 10. But we additionally require that the experimental and observational data share a common conditional distribution of the potential long-term outcome. This additional assumption ensures that the bridge functions defined in terms of the observational data distribution can also be used to identify the average long-term treatment effect over the experimental data distribution.

Finally, we note that the selection bridge functions can be used to identify more general parameters than the average treatment effects considered so far.

Corollary 3. Suppose Assumptions 1, 4, 9 and 10 and the assumptions in Corollary 1 condition 2 hold. Then for any function q_0 that satisfies Equation (12) or Equation (13), and any transformation $r: \mathcal{Y} \mapsto \mathbb{R}$, we have

$$\mathbb{E}\left[r(Y(a))\mid G=O\right] = \mathbb{E}\left[\frac{\mathbb{P}\left(G=E\mid A=a\right)\mathbb{P}\left(G=O\mid X\right)}{\mathbb{P}\left(G=O\mid A=a\right)\mathbb{P}\left(G=E\mid X\right)} \frac{\mathbb{I}\left[A=a\right]}{\mathbb{P}\left(A=a\mid X,G=E\right)} \times q_{0}\left(S_{2},S_{1},A,X\right)r(Y)\mid G=O\right].$$

In particular, when applying Corollary 3 to the indicator function $r(\cdot) = \mathbb{I}\left[\cdot \leq y\right]$ for all $y \in \mathcal{Y}$, we can identify the entire distribution of the counterfactual long term outcome Y(a).

D.2 Estimation

We can again leverage the doubly robust identification strategy in Theorem 8 to estimate the average long-term treatment effect. This involves some nuisance functions/parameters $\eta^* = (\eta_1^*, \eta_2^*, \dots, \eta_7^*)$:

$$\eta_1^*(S_3, S_2, A, X) = h_0(S_3, S_2, A, X), \quad \eta_2^*(X) = \left\{ \mathbb{E} \left[h_0(S_3, S_2, A, X) \mid A = a, X, G = E \right] : a = 0, 1 \right\}, \\
\eta_3^*(S_2, S_1, A, X) = q_0(S_2, S_1, A, X), \quad \eta_4^*(X) = \left\{ \mathbb{P} \left(A = a \mid X, G = E \right) : a = 0, 1 \right\}, \\
\eta_5^*(X) = \frac{\mathbb{P} \left(G = O \mid X \right)}{\mathbb{P} \left(G = E \mid X \right)}, \quad \eta_6^* = \frac{\mathbb{P} \left(G = E \mid A = a \right)}{\mathbb{P} \left(G = O \mid A = a \right)}, \quad \eta_7^* = \frac{\mathbb{P} \left(G = E \right)}{\mathbb{P} \left(G = O \right)}.$$

According to Theorem 8, once we know these nuisance functions/parameters, we immediately have

$$\tau = \sum_{a \in \{0,1\}} (-1)^{1-a} \left\{ \mathbb{E} \left[\phi_1(Y, S, a, X; \eta^*) \mid G = O \right] + \mathbb{E} \left[\phi_2(Y, S, a, X; \eta^*) \mid G = E \right] + \mathbb{E} \left[\phi_3(Y, S, a, X; \eta^*) \mid G = O \right] \right\},$$
(22)

where

$$\phi_{1}(Y, S, a, X; \eta^{*}) = \bar{h}_{E,0}(a, X) = \mathbb{E} \left[h_{0}(S_{3}, S_{2}, a, X) \mid A = a, X, G = E \right]$$

$$\phi_{2}(Y, S, a, X; \eta^{*}) = \frac{\mathbb{P} (G = E) \mathbb{P} (G = O \mid X)}{\mathbb{P} (G = O) \mathbb{P} (G = E \mid X)} \frac{\mathbb{I} [A = a]}{\mathbb{P} (A = a \mid X, G = E)} \left(h_{0}(S_{3}, S_{2}, a, X) - \bar{h}_{E,0}(a, X) \right)$$

$$\phi_{3}(Y, S, a, X; \eta^{*}) = \frac{\mathbb{P} (G = E \mid A = a) \mathbb{P} (G = O \mid X)}{\mathbb{P} (G = O \mid A = a) \mathbb{P} (G = E \mid X)} \frac{\mathbb{I} [A = a]}{\mathbb{P} (A = a \mid X, G = E)}$$

$$\times q_{0}(S_{2}, S_{1}, a, X) (Y - h_{0}(S_{3}, S_{2}, A, X)).$$

Again, we can follow Section 5 to construct an average long-term treatment effect estimator by plugging estimates in place of unknown nuisances above. In particular, we can use a cross-fitting procedure similar to Definition 1. Namely, we first split the two datasets into multiple folds, and for each fold, we estimate the treatment effect with plug-in nuisance estimates trained on the out-of-fold data, and finally average all treatment effect estimates across different folds. We denote the resulting cross-fitted treatment effect estimator as $\hat{\tau}$.

In the following lemma, we prove that the doubly robust equation above satisfies the so-called *Neyman orthogonality* property, which means that the corresponding treatment effect estimator is insensitive to estimation errors of the nuisance functions/parameters.

Lemma 3. The estimating equation implied by Equation (22) satisfies the Neyman Orthogonality property, namely, the pathwise derivative of the following map at η^* along any feasible direction is equal to 0:

$$\eta \mapsto \sum_{a \in \{0,1\}} (-1)^{1-a} \{ \mathbb{E} \left[\phi_1(Y, S, a, X; \eta) \mid G = O \right] + \mathbb{E} \left[\phi_2(Y, S, a, X; \eta) \mid G = E \right] + \mathbb{E} \left[\phi_3(Y, S, a, X; \eta) \mid G = O \right] \}.$$
 (23)

The Neyman orthogonality property plays a central role in the recent debiased machine learning literature [e.g., Chernozhukov et al., 2019]. It protects the estimator of primary parameters from the errors in estimating nuisance parameters, so that even when the nuisance estimators have slow

convergence rates (e.g., nonparametric machine learning estimators), the final estimator is still \sqrt{n} -consistent and asymptotically normal. In the following theorem, we show that this is the case for our cross-fitted average long-term treatment effect estimator.

Theorem 10. Suppose Assumptions 1, 4 to 7, 9 and 10 hold, and the nuisance estimator for every function in η^* converges to the truth at $o_{\mathbb{P}}(n^{-1/4})$ rate in terms of its root mean squared error. Then

$$\sqrt{n}(\hat{\tau} - \tau) \rightsquigarrow \mathcal{N}(0, \sigma^2),$$

where

$$\begin{split} \sigma^2 &= (1+\lambda) \mathbb{E} \left[(\phi_1(Y,S,1,X;\eta^*) - \phi_1(Y,S,0,X;\eta^*) - \tau)^2 \mid G = O \right] \\ &+ \frac{1+\lambda}{\lambda} \mathbb{E} \left[(\phi_2(Y,S,1,X;\eta^*) - \phi_2(Y,S,0,X;\eta^*))^2 \mid G = E \right] \\ &+ (1+\lambda) \mathbb{E} \left[(\phi_3(Y,S,1,X;\eta^*) - \phi_3(Y,S,0,X;\eta^*))^2 \mid G = O \right]. \end{split}$$

In Theorem 10, we show that as long as the nuisance functions are consistently estimated at $o_{\mathbb{P}}(n^{-1/4})$ rate, the cross-fitted treatment effect estimator $\hat{\tau}$ is \sqrt{n} -consistent and has an asymptotic normal distribution.

E Additional Extensions

In this section, we extend our identification results to more settings. For simplicity, we focus on extending the first identification strategy in Theorem 1.

E.1 Pre-treatment Outcomes

In the main text, the short-term outcomes $S = (S_1, S_2, S_3)$ are all post-treatment outcomes. In this part, we let part of the outcomes be pre-treatment.

We first consider the setting where S_1 is pre-treatment but S_2, S_3 are post-treatment. Below we modify Assumptions 1 to 4 accordingly.

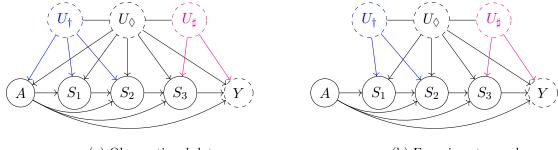
Assumption 13 (Pre-treatment S_1). Suppose the following hold for $a \in \{0, 1\}$:

- 1. On the observational data, we have $(Y(a), S_3(a), S_2(a)) \perp A \mid S_1, U, X, G = O$ and $0 < \mathbb{P}(A = 1 \mid S_1, U, X, G = O) < 1$ almost surely.
- 2. On the experimental data, we have $(Y(a), S_3(a), S_2(a), U) \perp A \mid S_1, X, G = E$ and $0 < \mathbb{P}(A = 1 \mid S_1, X, G = O) < 1$ almost surely.
- 3. The external validity $(S_3(a), S_2(a), U) \perp G \mid S_1, X$ and overlap

$$\frac{p(S_1, U, X \mid A = a, G = E)}{p(S_1, U, X \mid A = a, G = O)} < \infty, \quad almost \ surely.$$

4. The sequential structure $(Y(a), S_3(a)) \perp S_1 \mid S_2(a), U, X, G = O$.

Note that Assumption 13 consider the most general setting: we allow the treatment assignments in the observational and experimental data to depend on pre-treatment outcomes S_1 , and also allow the distribution of S_1 to be different on the two datasets. Now we extend our identification strategy to this setting.



(a) Observational data.

(b) Experiment sample.

Figure 7: unobserved confounders $(U_{\sharp}, U_{\dagger})$ that can be ignored. Additional covariates X can be present but we do not draw them to avoid cluttering the graphs.

Corollary 4. Suppose conditions in Assumption 13, the completeness condition in Assumption 5 condition 2 and Assumption 6 hold. Then the average long-term treatment effect is identifiable: for any function h_0 that satisfies Equation (9),

$$\tau = \mathbb{E}\left[\mathbb{E}\left[h_0(S_3, S_2, A, X) \mid S_1, A = 1, X, G = E\right] \mid G = O\right] - \mathbb{E}\left[\mathbb{E}\left[h_0(S_3, S_2, A, X) \mid S_1, A = 0, X, G = E\right] \mid G = O\right].$$
(24)

The identification strategy in Equation (24) is very similar to Equation (20). Equation (24) essentially augments the covariates X with the pre-treatment outcomes S_1 .

Similarly, we can also consider the setting where both S_1 and S_2 are pre-treatment.

Assumption 14 (Pre-treatment (S_1, S_2)). Suppose the following hold for $a \in \{0, 1\}$:

- 1. On the observational data, we have $(Y(a), S_3(a)) \perp A \mid S_2, S_1, U, X, G = O$ and $0 < \mathbb{P}(A = 1 \mid S_2, S_1, U, X, G = O) < 1$ almost surely.
- 2. On the experimental data, we have $(Y(a), S_3(a), U) \perp A \mid S_2, S_1, X, G = E$ and $0 < \mathbb{P}(A = 1 \mid S_2, S_1, X, G = O) < 1$ almost surely.
- 3. The external validity $(S_3(a), U) \perp G \mid S_2, S_1, X$ and overlap

$$\frac{p(S_2, S_1, U, X \mid A = a, G = E)}{p(S_2, S_1, U, X \mid A = a, G = O)} < \infty, \quad almost \ surely.$$

4. The sequential structure $(Y(a), S_3(a)) \perp S_1 \mid S_2, U, X, G = O$.

We can analogously identify the long-term average treatment effect

Corollary 5. Suppose conditions in Assumption 14, the completeness condition in Assumption 5 condition 2 and Assumption 6 hold. Then the average long-term treatment effect is identifiable: for any function h_0 that satisfies Equation (9),

$$\tau = \mathbb{E}\left[\mathbb{E}\left[h_0(S_3, S_2, A, X) \mid S_2, S_1, A = 1, X, G = E\right] \mid G = O\right] - \mathbb{E}\left[\mathbb{E}\left[h_0(S_3, S_2, A, X) \mid S_2, S_1, A = 0, X, G = E\right] \mid G = O\right].$$
 (25)

E.2 Partial Confounding Adjustments

In the main text, the unobseved variables U stand for all unobserved confounders that can possibly affect the treatment, the short-term outcomes, the long-term outcome, or any subset of them (see Figure 3). The identification strategies in Section 4 require the short-term outcomes (S_1, S_3) to be sufficiently rich relative to all of the unobserved confounders. In this part, we show that actually we do not need to use the short-term outcomes to handle all such unobserved confounders. Instead, we can achieve identification under lower requirements for the short-term outcomes, still using the same identification strategies.

In Figure 7, we plot three different types of unobserved confounders: confounders U_{\diamond} can affect any of (Y, S_3, S_2, S_1, A) , confounders U_{\dagger} can affect (S_2, S_1, A) but not (S_3, Y) , while confounders U_{\sharp} can affect (S_3, Y) but not (S_2, S_1, A) . Naively, one can view $U = (U_{\diamond}, U_{\dagger}, U_{\sharp})$ and argue identifiability following any of Theorems 1 to 3. This would require the short-term outcomes (S_1, S_3) to be rich enough relative to all of $(U_{\diamond}, U_{\dagger}, U_{\sharp})$. Now we show that this is not necessary. Instead, we need (S_1, S_3) to be rich enough relative to only U_{\diamond} , but not necessarily $(U_{\dagger}, U_{\sharp})$.

We first extend Assumptions 1 to 4 to the current setting, by substituting U_{\diamond} for U in these previous assumptions.

Assumption 15. Assume the following conditions hold for any $a \in \{0,1\}$:

1.
$$(Y(a), S_3(a)) \perp A \mid S_2(a), U_{\diamond}, X, G = O \text{ and } 0 < \mathbb{P}(A = 1 \mid U_{\diamond}, X, G = O) < 1 \text{ almost surely.}$$

2.
$$(S_2(a), U_{\diamond}) \perp A \mid X, G = E \text{ and } 0 < \mathbb{P}(A = 1 \mid X, G = E) < 1 \text{ almost surely.}$$

3. $(S_3(a), S_2(a), U_{\diamond}) \perp G \mid X$, and

$$\frac{p(U_{\diamond}, X \mid A = a, G = E)}{p(U_{\diamond}, X \mid A = a, G = O)} < \infty.$$

4.
$$(Y(a), S_3(a)) \perp S_1(a) \mid S_2(a), U_{\diamond}, X, G = O.$$

It is easy to verify that the current setting depicted in Figure 7 can satisfy Assumption 15. Moreover, below we modify the completeness condition in Assumption 5 condition 2 and the outcome bridge function assumption in Assumption 6.

Assumption 16. 1. For any $s_2 \in \mathcal{S}_2$, $a \in \{0,1\}$, $x \in \mathcal{X}$,

if
$$\mathbb{E}[q(U_{\diamond}) \mid S_1, S_2 = s_2, A = a, X = x, G = O] = 0$$
 holds almost surely,

then $g(U_{\diamond}) = 0$ almost surely.

2. There exists an outcome bridge function $h_0: S_3 \times S_2 \times A \times X \to \mathbb{R}$ such that

$$\mathbb{E}[Y \mid S_2, A, U_{\diamond}, X, G = O] = \mathbb{E}[h_0(S_3, S_2, A, X) \mid S_2, A, U_{\diamond}, X, G = O]. \tag{26}$$

In Assumption 16(a), we assume a partial completeness condition, which only require the short-term outcomes S_1 to be rich enough relative to U_{\diamond} . In Assumption 16(b), we only require the bridge function to capture the unmeasured confounding due to U_{\diamond} . This is possible when the short-term outcomes S_3 are rich enough relative to U_{\diamond} . Importantly, we do not need S_1, S_3 to be rich enough relative to $(U_{\diamond}, U_{\dagger}, U_{\dagger})$ together.

Then we show that the long-term average treatment effect can be identified according to the equation we derived in Corollary 1. This means actually the same identification strategy still works under lower requirements on the short-term outcomes.

Corollary 6. Suppose Assumptions 15 and 16 hold. Then the average long-term treatment effect is identifiable: for any function h_0 that satisfies Equation (9), Equation (20) in Corollary 1 holds.

E.3 Relaxing the External Validity Assumption

In Section 6 Assumption 9, we assumed the external validity condition that the distributions of the unobserved confounders U on the two datasets, conditional on the covariates X, are identical. In this section, we show that this assumption can be weakened, provided that S_1 and S_2 are both pre-treatment outcomes. Specifically, we assume the following condition.

Assumption 17. Suppose that for any $a \in \{0, 1\}$,

$$G \perp S_3(a) \mid S_2, S_1, U, X.$$

Assumption 17 imposes that the distributions of the potential short-term outcome $S_3(a)$ are identical on the two datasets, conditional on the pre-treatment outcomes S_2, S_1 , the unobserved confounders U, and the observed covariates X. Importantly, this assumption is weaker than the condition $G \perp (S_3(a), U) \mid S_2, S_1, X$, allowing for distribution shift of the unobserved confounders U. To handle the lack of external validity, we again view the short-term outcomes as proxies for the unobserved confounders. Specifically, we rely on the following external validity bridge function.

Assumption 18 (External validity bridge function). There exists an external validity bridge function $\tilde{q}: \mathcal{S}_2 \times \mathcal{S}_1 \times \mathcal{A} \times \mathcal{X} \to \mathbb{R}$ defined as follows:

$$\frac{p(S_2, U, X \mid A, G = O)}{p(S_2, U, X \mid A, G = E)} = \mathbb{E}[\tilde{q}(S_2, S_1, A, X) \mid S_2, A, U, X, G = E]$$
(27)

The external validity bridge function in Assumption 18 Equation (27) is very similar to the selection bridge function in Assumption 7 Equation (11). There are only two differences: one is that the left hand side of Equation (27) is the reciprocal of the left hand side of Equation (11), and the other is that the right hand side of Equation (27) involves a conditional expectation over the experimental data rather than the observational data. We will show that the external validity bridge function can adjust for the discrepancy in the distributions of unobserved confounders between the two datasets. Since the external validity bridge function in Equation (27) is defined in terms of unobserved confounders, we cannot directly use this definition to learn an external validity bridge function. Instead, we give an alternative formulation that involves only the observed variables.

Lemma 4. Assume Assumption 14 conditions 1, 2, 4, Assumption 17, and the completeness condition in Assumption 5 condition 2. Then any function \tilde{q} that satisfies

$$\frac{p(S_3, S_2, X \mid A, G = O)}{p(S_3, S_2, X \mid A, G = E)} = \mathbb{E}[\tilde{q}(S_1, S_2, X, A) \mid S_2, X, S_3, A, G = E]$$

is also a valid external validity bridge function in the sense of Equation (27).

In the theorem below, we further show that the the average treatment effect can be identified by any external validity function and any outcome bridge function.

Theorem 11. Assume the assumptions in Lemma 4, Assumption 6, and $(S_2, S_1, U, X) \perp A \mid G = E \text{ hold. Let } \tilde{q} \text{ and } h \text{ be any functions that satisfy Equation (27) and Equation (9) respectively. Then for <math>a \in \{0,1\}$, we have

$$\mathbb{E}[Y(a) \mid S_2, X, G = O] = m(S_2, a, X) \frac{p(S_2, X \mid G = E)}{p(S_2, X \mid G = O)},$$

where

$$m(S_2, a, X) := \mathbb{E}\left[\mathbb{E}[h(S_3, S_2, X, A) \mid S_2, S_1, X, G = E, A = a] \sum_{a'} \mathbb{P}\left(A = a' \mid G = O\right) \tilde{q}(S_2, S_1, X, a') \mid S_2, X, A = a, G = E\right].$$

Moreover, we have

$$\tau = \mathbb{E} [m(S_2, X, 1) - m(S_2, X, 0) \mid G = E].$$

F Additional Results for Numerical Studies

F.1 Additional Details for Section 7.1

In the following proposition, we justify the sampling probability function described in Section 7.1.1.

Proposition 3. Let (Z_1, Z_2, A) be a random vector with $(Z_1, Z_2) \perp A$ and $A \in \{0, 1\}$. Let $G \in \{0, 1\}$ be a binary random variable such that $G \perp Z_2 \mid Z_1$ and

$$\mathbb{P}(G=1 \mid Z_1, A=1) \, \mathbb{P}(A=1) + \mathbb{P}(G=1 \mid Z_1, A=0) \, \mathbb{P}(A=0) \equiv C,$$

where C is a positive constant. Then the probability density of (Z_1, Z_2) satisfies that

$$p(z_1, z_2 \mid G = 1) \equiv p(z_1, z_2), \quad \forall z_1, z_2.$$

We can let Z_1 be the education level U, Z_2 be other covariates and the potential short-term outcomes, A be the GAIN treatment assignment, and G be the indicator for whether being selected into the observational dataset \mathcal{D}_O . Then Proposition 3 means that the subsampling procedure does not change the distribution of latent confounders, covariates, and potential short-term outcomes. This explains why the subsampling is not against Assumption 3.

F.2 Additional Results for Section 7.1

We further use data to probe the plausibility of the Assumptions 6 and 7 in our GAIN case study. According to Example 1, in a discrete setting, Assumptions 6 and 7 hold when certain conditional probability matrices have full column rank. We note that the outcomes in the GAIN dataset empirical example are all discrete, so we we design some heurstic assessments here to shed some light on Assumptions 6 and 7 in the empirical study. Based on the GAIN dataset, we estimate the conditional probability matrices $P(\mathbf{S_1} \mid S_2 = s_2, A = a, \mathbf{U})$ and $P(\mathbf{S_3} \mid S_2 = s_2, A = a, \mathbf{U})$ by their empirical frequencies (we do not condition on X since this is difficult noting that X is multi-dimensional and some components are continuous), for $s_2 \in \{(0,0),(1,0),(0,1),(1,1)\}$ and $a \in \{0,1\}$. We vary the dimension of S_1 and S_3 (i.e., the number of employment status variables included in S_1, S_3 respectively) from 2 to 6 while fixing the dimension of S_2 as 2 (the number we used in our original numerical study). The smallest singular values of the corresponding empirical probability matrices are calculated and shown in Table 3. We can observe that the smallest singular value gets consistently larger as dimension of S_1 and S_3 increases, unless when the smallest singular value is already sufficiently large. This heuristically suggests that Assumptions 6 and 7 are more likely to hold if we incorporate more short-term outcomes in S_1, S_3 , validating our high level intuitions discussed above.

$\dim(S_1)$	s_2	$P(\mathbf{S}_1 \mid S_2 = s_2, A = 0, \mathbf{U})$	$P(\mathbf{S}_3 \mid S_2 = s_2, A = 0, \mathbf{U})$	$P(\mathbf{S}_1 \mid S_2 = s_2, A = 1, \mathbf{U})$	$P(\mathbf{S}_3 \mid S_2 = s_2, A = 1, \mathbf{U})$
2	(0,0)	0.016	0.012	0.002	0.007
	(1,0)	0.015	0.068	0.059	0.018
	(0,1)	0.014	0.019	0.002	0.024
	(1,1)	0.026	0.038	0.010	0.020
	(0,0)	0.023	0.023	0.018	0.009
4	(1,0)	0.190	0.125	0.072	0.055
4	(0,1)	0.123	0.185	0.067	0.032
	(1,1)	0.063	0.084	0.036	0.028
	(0,0)	0.028	0.020	0.020	0.014
6	(1,0)	0.134	0.145	0.068	0.053
	(0,1)	0.151	0.165	0.070	0.083
	(1,1)	0.084	0.072	0.040	0.037

Table 3: List of smallest singular values of the empirical estimates of the conditional probability matrices $P(\mathbf{S}_1 \mid S_2 = s_2, A = a, \mathbf{U})$ and $P(\mathbf{S}_3 \mid S_2 = s_2, A = a, \mathbf{U})$ for $s_2 = (0, 0), \dots, (1, 1), a = 0, 1$ with different dimension of S_1 and S_3 . Here we keep throughout the dimension of S_1 and S_3 to be the same. The dimension of S_1 corresponds to the number of quarters included in the surrogate. Here $\dim(S_1) = 2$ means that S_1, S_2 and S_3 take the employment status of 1 - 2-th quarters, 3 - 4-th quarters and 5 - 6-th quarters after the treatment respectively; $\dim(S_1) = 4$ means the three surrogates take 1 - 4-th quarters, 5 - 6-th quarters and 7 - 10-th quarters respectively; and $\dim(S_1) = 6$ means the three surrogates take 1 - 6-th quarters, 7 - 8-th quarters and 9 - 14-th quarters respectively.

Of course, in Table 3 some smallest singular values are indeed fairly small, posing threats to Assumptions 6 and 7. However, we find that across all these settings, the performance of our proposed estimator is overall stable and it is significantly better than the existing state-of-art estimator in Athey et al. [2020]. These results show potential benefit of using our method to account for general unobserved confounding, even if our assumptions may not necessarily hold exactly.

Specifically, we already show the performance of our estimator in Section 7.1 Table 1 for the setting $\dim(S_1) = \dim(S_2) = \dim(S_3) = 2$. In Table 4, we generate the data in the same way as in Table 1, but with (S_1, S_2, S_3) as the employment status in the 1 – 4-th quarters, 5 – 6-th quarters, and 7 – 10-th quarters. In other words, we keep $\dim(S_2) = 2$ but increase the dimension of both S_1 and S_3 to $\dim(S_2) = \dim(S_3) = 4$. In Table 5, we set (S_1, S_2, S_3) as the employment status in 1 – 6-th quarters, 7 – 8-th quarters and 9 – 14-th quarters, i.e., we increase the dimension of S_1 and S_3 to $\dim(S_2) = \dim(S_3) = 6$. Apparently, with ridge regularization (namely the ".33", ".67" and "1" columns), our estimator is still consistently better than Athey et al. [2020] by a large margin, showing that the performance of our estimator is stable with respect to the number of quarters in surrogate construction. Interestingly, with the existence of ridge regularization, our estimator can perform slightly worse as we increase the dimension of surrogates, which may be due to the non-uniqueness of bridge functions. When the ridge regularization does not exist (namely the "0" column), our estimator can be quite unstable, sometimes even worse than the naive estimator. Such phenomenon has also been observed Table 1.

F.3 Implementation details of the minimax approach in Section 7.2

In this section, we provide implementation details of the minimax approach in Section 7.2. To

-		$\hat{ au}_{ ext{OTC}}$					$\hat{ au}_{ ext{S}}$	EL			$\hat{ au}_{ ext{DI}}$	3	Athey	Naive		
$\overline{\eta}$		0	.33	.67	1	0	.33	.67	1	0	.33	.67	1	NR	CV	
0	MAE	-560	74	76	77	81	78	83	85	-465	78	81	81	13	33	0.053
	Med	-560	74	76	77	81	78	83	85	-465	78	81	81	13	33	0.053
0.2	MAE	-416	73	74	74	78	78	78	78	-363	77	79	79	21	31	0.059
0.2	Med	33	72	74	74	78	77	77	78	41	77	78	79	21	31	0.059
0.4	MAE	-150	71	72	72	75	75	75	75	-151	76	77	77	27	28	0.067
0.4	Med	43	71	72	72	75	76	75	76	45	76	77	77	28	29	0.067
0.6	MAE	-577	68	68	68	72	72	72	72	-574	73	74	74	34	25	0.076
0.0	Med	44	68	68	68	73	72	73	72	47	74	74	74	34	25	0.076
0.8	MAE	-685	63	63	63	67	67	67	67	-652	69	70	69	37	21	0.088
0.8	Med	534	62	62	62	67	67	67	67	37	69	69	69	37	21	0.088
1	MAE	-135	63	63	63	68	68	68	68	-139	70	70	70	38	18	0.095
1	Med	35	63	63	62	69	69	69	68	36	70	70	71	39	18	0.095
1.2	MAE	-237	62	62	62	68	68	68	68	-213	70	70	70	38	15	0.104
1.2	Med	28	62	62	62	69	69	69	68	31	71	71	71	39	15	0.104
1.4	MAE	-241	61	60	59	69	69	68	68	-254	70	70	70	37	11	0.115
1.4	Med	11	61	61	60	71	71	71	70	14	73	73	72	36	11	0.115
1.6	MAE	-271	59	58	57	68	68	68	68	-292	70	70	69	37	10	0.124
	Med	4	60	59	58	71	71	71	70	5	73	72	72	36	10	0.124

Table 4: Same as Table 1, but with S_1, S_2, S_3 taking the quarters $1-4, \, 5-6$ and 7-10.

		$\hat{ au}_{ ext{OTC}}$				$\hat{ au}_{ m SEL}$					l	Athey	Naive			
$\overline{\eta}$		0	.33	.67	1	0	.33	.67	1	0	.33	.67	1	NR	CV	
0	MAE	-1030	74	72	71	75	73	75	72	-1010	77	76	75	17	39	0.053
	Med	-1030	74	72	71	75	73	75	72	-1010	77	76	75	17	39	0.053
0.2	MAE	-9723	72	70	69	75	75	74	74	-9789	76	75	74	23	39	0.059
	Med	-184	72	70	69	75	75	74	74	-184	76	75	74	23	38	0.059
0.4	MAE	-1102	70	68	67	74	73	73	73	-1113	74	73	73	29	38	0.067
	Med	-152	70	68	67	74	73	73	73	-152	75	74	73	29	38	0.067
0.6	MAE	-861	67	65	64	72	71	71	71	-876	72	71	70	35	36	0.076
0.0	Med	-149	67	65	64	72	72	71	71	-152	72	72	71	35	36	0.076
0.8	MAE	-8496	62	60	59	68	68	67	67	-8513	69	67	67	37	32	0.088
	Med	-94	61	59	58	68	68	67	66	-92	69	67	67	37	32	0.088
1	MAE	-640	59	58	57	68	68	67	67	-645	67	66	66	38	30	0.095
	Med	-111	59	58	57	69	68	67	67	-112	67	66	66	38	29	0.095
1.2	MAE	-459	57	55	55	68	67	67	66	-467	65	65	64	37	26	0.104
1.4	Med	-96	57	56	55	69	68	68	68	-102	66	66	65	37	26	0.104
1.4	MAE	-2157	53	51	51	66	66	66	65	-2210	63	63	63	33	21	0.115
	Med	-96	53	52	51	69	68	67	67	-101	65	64	64	33	21	0.115
1.6	MAE	-683	50	49	48	66	65	65	64	-714	62	62	62	31	17	0.124
	Med	-73	50	49	48	68	67	66	66	-78	64	63	63	31	17	0.124

Table 5: Same as Table 1, but with S_1, S_2, S_3 taking the quarters 1-6, 7-8 and 9-14.

construct the outcome bridge function, we set the outer minimization function class as a neural network class with four layers. For $\dim(X) = 10, 15, 20$, we choose the number of neurons in first and second hidden layers to be 50 and 10, respectively; for $\dim(X) = 5$, we change the number of neurons in the first hidden layer to 30. We set the momentum, learning rate, number of epochs of the neural network optimizer to be 0.95, 0.0002 and 40, respectively; and set the size of each batch to be 1 / 10 of the total sample size. We use the ReLU activation function for the first three layers and set the activation function for the last layer as a linear function. For the inner maximization function class, we set it a RKHS class with a product radial basis function kernel. To construct the selection bridge function, we use a similar neural architecture as in the outcome bridge function construction, except that we set the activation function in the last layer as a softplus activation function; for the inner maximization function class, we use a RKHS with a linear kernel.

G Proofs

G.1 Supporting Lemmas

Lemma 5. Under Assumptions 1 to 3, we have

$$(S_3, S_2) \perp G \mid A, U, X. \tag{28}$$

Proof. For any $a \in \mathcal{A}$, $s_3 \in \mathcal{S}_3$, $s_2 \in \mathcal{S}_2$ and $g \in \{E, O\}$, we have

$$\begin{split} p_{S_3,S_2}(s_3,s_2 \mid U,X,A=a,G=g) &= p_{S_3(a),S_2(a)}(s_3,s_2 \mid U,X,A=a,G=g) \\ &= p_{S_3(a),S_2(a)}(s_3,s_2 \mid U,X,G=g) \\ &= p_{S_3(a),S_2(a)}(s_3,s_2 \mid U,X) \\ &= p_{S_3,S_2}(s_3,s_2 \mid U,X,A=a), \end{split}$$

where the second equation follows from Assumptions 1 and 2 and the third equation follows from Assumption 3. \Box

Lemma 6. Under Assumptions 1 and 4, we have

$$(Y, S_3) \perp S_1 \mid S_2, A, U, X, G = O.$$
 (29)

Proof. For any $a \in \mathcal{A}, s \in \mathcal{S}_2$ and any bounded continous functions $f : \mathcal{Y} \times \mathcal{S}_3 \to \mathbb{R}$ and $g : \mathcal{S}_1 \to \mathbb{R}$, we have

$$\mathbb{E}\left[f(Y,S_3)g(S_1) \mid S_2 = s, U, X, A = a, G = O\right]$$

$$= \mathbb{E}\left[f(Y(a), S_3(a))g(S_1(a)) \mid S_2(a) = s, U, X, A = a, G = O\right]$$

$$= \mathbb{E}\left[f(Y(a), S_3(a))g(S_1(a)) \mid S_2(a) = s, U, X, G = O\right]$$

$$= \mathbb{E}\left[f(Y(a), S_3(a)) \mid S_2(a) = s, U, X, G = O\right] \mathbb{E}\left[g(S_1(a)) \mid S_2(a) = s, U, X, G = O\right]$$

$$= \mathbb{E}\left[f(Y, S_3) \mid S_2 = s, U, X, A = a, G = O\right] \mathbb{E}\left[g(S_1) \mid S_2 = s, U, X, A = a, G = O\right],$$

where the second equation follows from Assumption 1, the third equation follows from Equation (6) in Assumption 4, and the fourth equation again follows from Assumption 1. \Box

Lemma 7. Under Assumption 3, for any $a \in A$, the following holds almost surely:

$$\frac{p(S_2, U, X \mid A = a, G = E)}{p(S_2, U, X \mid A = a, G = O)} = \frac{p(U, X \mid A = a, G = E)}{p(U, X \mid A = a, G = O)} < \infty$$
(30)

Proof. This is proved by noting that

$$\begin{split} \frac{p(S_2, U, X \mid A = a, G = E)}{p(S_2, U, X \mid A = a, G = O)} &= \frac{p(S_2(a), U, X \mid A = a, G = E)}{p(S_2(a), U, X \mid A = a, G = O)} \\ &= \frac{p(S_2(a) \mid U, X, A = a, G = E)}{p(S_2(a) \mid U, X, A = a, G = O)} \frac{p(U, X \mid A = a, G = E)}{p(U, X \mid A = a, G = O)} \\ &= \frac{p(U, X \mid A = a, G = E)}{p(U, X \mid A = a, G = O)} < \infty. \end{split}$$

where the last equation follows from Equation (4) in Assumption 3.

G.2 Proofs for Section 4.1

Proof for lemma 1. In lemma 6, we already proved that Assumptions 1 and 4 imply

$$(Y, S_3) \perp S_1 \mid S_2, A, U, X, G = O.$$

Therefore, for any function $h_0(S_3, S_2, A, X)$, we have

$$\mathbb{E}[Y \mid S_2, S_1, A, G = O] = \mathbb{E}[\mathbb{E}[Y \mid S_2, S_1, A, U, X, G = O] \mid S_2, S_1, A, X, G = O]$$
$$= \mathbb{E}[\mathbb{E}[Y \mid S_2, A, U, X, G = O] \mid S_2, S_1, A, X, G = O],$$

and

$$\mathbb{E} [h_0(S_3, S_2, A, X) \mid S_2, S_1, A, X, G = O]$$

$$= \mathbb{E} [\mathbb{E} [h_0(S_3, S_2, A, X) \mid S_2, S_1, A, U, X, G = O] \mid S_2, S_1, A, X, G = O]$$

$$= \mathbb{E} [\mathbb{E} [h_0(S_3, S_2, A, X) \mid S_2, A, U, X, G = O] \mid S_2, S_1, A, X, G = O].$$

Therefore, for any $h_0(S_3, S_2, A, X)$ that satisfies eq. (7), we have

$$0 = \mathbb{E} [Y - h_0(S_3, S_2, A, X) \mid S_2, S_1, A, X, G = O]$$

= $\mathbb{E} [\mathbb{E} [Y - h_0(S_3, S_2, A, X) \mid S_2, A, U, X, G = O] \mid S_2, S_1, A, X, G = O].$

It follows from the completeness condition in Assumption 5 condition 2 that

$$\mathbb{E}[Y - h_0(S_3, S_2, A, X) \mid S_2, A, U, X, G = O] = 0,$$

Namely, any function $h_0(S_3, S_2, A, X)$ that satisfies eq. (9) is a valid outcome bridge function satisfying eq. (7).

Proof for theorem 1. According to Lemma 1, any function h_0 that solves Equation (9) also satisfies Equation (7). Thus we only need to show that for any function h_0 that solves Equation (7), we have $\mu(a) = \mathbb{E}[h_0(S_3, S_2, A, X) \mid A = a, G = E]$. This is proved as follows:

$$\mathbb{E} [h_0(S_3, S_2, A, X) \mid A = a, G = E]$$

$$= \mathbb{E} [\mathbb{E} [h_0(S_3, S_2, a, X) \mid S_2, A = a, U, X, G = E] \mid A = a, G = E]$$

$$= \mathbb{E} [\mathbb{E} [h_0(S_3, S_2, a, X) \mid S_2, A = a, U, X, G = O] \mid A = a, G = E]$$

$$= \mathbb{E} [\mathbb{E} [Y \mid S_2, A = a, U, X, G = O] \mid A = a, G = E]$$

$$= \mathbb{E} [\mathbb{E} [Y(a) \mid S_2(a), U, X, G = O] \mid A = a, G = E]$$

$$= \mathbb{E} [\mathbb{E} [Y(a) \mid S_2(a), U, X, G = O] \mid G = E]$$

$$= \mathbb{E} [\mathbb{E} [Y(a) \mid S_2(a), U, X, G = O] \mid G = O] = \mathbb{E} [Y(a) \mid G = O] = \mu(a),$$

where the second equation uses the fact that $G \perp S_3 \mid S_2, A = a, U, X$ (see Equation (28) in Lemma 5) and Equation (30) in Lemma 7, the third equation uses the definition of the outcome bridge function, the fourth equation uses the fact that $Y(a) \perp A \mid S_2(a), U, X, G = O$ according to Assumption 1, the fifth uses the fact that $(S_2(a), U, X) \perp A \mid G = E$ according to Assumption 2, and the sixth equation holds because $G \perp (S_2(a), U, X)$ in Assumption 3 and Equation (30) in Lemma 7.

G.3 Proofs for Section 4.2

Proof for Lemma 2. First note that

$$p(S_3, S_2, X \mid A, G = E)$$

$$= \int p(S_3 \mid S_2, A, U = u, X, G = E) p(S_2, u, X \mid A, G = E) du$$

$$= \int p(S_3 \mid S_2, A, U = u, X, G = O) p(S_2, u, X \mid A, G = E) du,$$

where the second equation follows from $S_3 \perp G \mid S_2, A, U, X$ that we prove in Lemma 5. Next, note that

$$\begin{split} &p(S_3, S_2, X \mid A, G = O) \mathbb{E} \left[q_0(S_2, S_1, A, X) \mid S_3, S_2, A, X, G = O \right] \\ &= p(S_3, S_2, X \mid A, G = O) \int p(u \mid S_3, S_2, A, X, G = O) \mathbb{E} \left[q_0(S_2, S_1, A, X) \mid S_3, S_2, A, U = u, X, G = O \right] \mathrm{d}u \\ &= p(S_3, S_2, X \mid A, G = O) \int p(u \mid S_3, S_2, A, X, G = O) \mathbb{E} \left[q_0(S_2, S_1, A, X) \mid S_2, A, U = u, X, G = O \right] \mathrm{d}u \\ &= \int p(S_3, S_2, u, X \mid A, G = O) \mathbb{E} \left[q_0(S_2, S_1, A, X) \mid S_2, A, U = u, X, G = O \right] \mathrm{d}u \\ &= \int p(S_3 \mid S_2, A, U = u, X, G = O) p(S_2, u, X \mid A, G = O) \mathbb{E} \left[q_0(S_2, S_1, A, X) \mid S_2, A, U = u, X, G = O \right] \mathrm{d}u \end{split}$$

where the second equation follows from $S_1 \perp S_3 \mid S_2, A, U, X, G = O$ that we prove in Lemma 6. Therefore, any function q_0 that satisfies Equation (12) must satisfy

$$\int p(S_3 \mid S_2, A, U = u, X, G = O) \Delta(S_2, A, u, X) du = 0,$$

where

$$\Delta(S_2, A, U, X) = p(S_2, U, X \mid A, G = E) - p(S_2, U, X \mid A, G = O) \mathbb{E} \left[q_0(S_2, S_1, A, X) \mid S_2, A, U, X, G = O \right].$$

By Bayes rule, this is equivalent to

$$P(S_3 \mid S_2, A, X, G = O) \mathbb{E}\left[\frac{\Delta(S_2, A, U, X)}{p(U \mid S_2, A, X, G = O)} \mid S_3, S_2, A, X, G = O\right] = 0.$$

According to assumption 5 condition 1, we have $\Delta(S_2, A, U, X) = 0$ almost surely. In other words, if q_0 satisfies Equation (12), then it must also satisfy Equation (11).

Lemma 8. Under assumptions in Lemma 2, Equation (12) is equivalent to Equation (13).

Proof. Note that Equation (12) is equivalent to

$$\begin{split} \mathbb{E}\left[q_{0}(S_{2}, S_{1}, A, X) \mid S_{3}, S_{2}, A, X, G = O\right] &= \frac{p(S_{3}, S_{2}, X \mid A, G = E)}{p(S_{3}, S_{2}, X \mid A, G = O)} \\ &= \frac{\mathbb{P}\left(G = E \mid S_{3}, S_{2}, A, X\right) \mathbb{P}\left(G = O \mid A\right)}{\mathbb{P}\left(G = O \mid S_{3}, S_{2}, A, X\right) \mathbb{P}\left(G = E \mid A\right)} \\ &= \frac{(1 - \mathbb{P}\left(G = O \mid S_{3}, S_{2}, A, X\right)) \mathbb{P}\left(G = O \mid A\right)}{\mathbb{P}\left(G = O \mid S_{3}, S_{2}, A, X\right) \mathbb{P}\left(G = E \mid A\right)}. \end{split}$$

It is equivalent to

$$\mathbb{P}(G = O \mid S_3, S_2, A, X) \mathbb{E}\left[\frac{\mathbb{P}(G = E \mid A)}{\mathbb{P}(G = O \mid A)}q_0(S_2, S_1, A, X) \mid S_3, S_2, A, X, G = O\right]$$

=1 - \mathbb{P}(G = O \cap S_3, S_2, A, X),

or

$$\mathbb{P}(G = O \mid S_3, S_2, A, X) \mathbb{E}\left[\frac{\mathbb{P}(G = E \mid A)}{\mathbb{P}(G = O \mid A)}q_0(S_2, S_1, A, X) + 1 \mid S_3, S_2, A, G = O\right] = 1.$$

The conclusion then follows straightforwardly.

Proof for Theorem 2. According to Lemma 2, any function q_0 that solves Equation (12) or Equation (13) must also satisfy Equation (11). Thus we only need to show that for any q_0 that solves Equation (11), we have

$$\mu(a) = \mathbb{E} [q(S_2, S_1, A, X)Y \mid A = a, G = O].$$

This is proved as follows:

$$\begin{split} &\mathbb{E}\left[q_{0}(S_{2},S_{1},A,X)Y\mid A=a,G=O\right] \\ &=\mathbb{E}\left[\mathbb{E}\left[q_{0}(S_{2},S_{1},A,X)Y\mid S_{2},A,U,X,G=O\right]\mid A=a,G=O\right] \\ &=\mathbb{E}\left[\mathbb{E}\left[q_{0}(S_{2},S_{1},A,X)Y\mid S_{2},A,U,X,G=O\right]\mid A=a,G=O\right] \\ &=\mathbb{E}\left[\mathbb{E}\left[q_{0}(S_{2},S_{1},A,X)\mid S_{2},A,U,X,G=O\right]\mathbb{E}\left[Y\mid S_{2},A,U,X,G=O\right]\mid A=a,G=O\right] \\ &=\mathbb{E}\left[\mathbb{E}\left[q_{0}(S_{2}(a),S_{1}(a),A,X)\mid S_{2}(a),A=a,U,X,G=O\right]\mid A=a,G=O\right] \\ &=\mathbb{E}\left[\mathbb{E}\left[q_{0}(S_{2},S_{1},A,X)\mid S_{2},A,U,X,G=O\right]\mathbb{E}\left[Y(a)\mid S_{2}(a),U,X,G=O\right]\mid A=a,G=O\right] \\ &=\mathbb{E}\left[\frac{p(S_{2},U,X\mid A,G=E)}{p(S_{2},U,X\mid A,G=O)}\mathbb{E}\left[Y(a)\mid S_{2}(a),U,X,G=O\right]\mid A=a,G=O\right] \\ &=\mathbb{E}\left[\frac{p(S_{2}(a),U,X\mid A=a,G=E)}{p(S_{2}(a),U,X\mid A=a,G=O)}\mathbb{E}\left[Y(a)\mid S_{2}(a),U,X,G=O\right]\mid A=a,G=O\right] \\ &=\mathbb{E}\left[\mathbb{E}\left[Y(a)\mid S_{2}(a),U,X,G=O\right]\mid A=a,G=E\right] \\ &=\mathbb{E}\left[\mathbb{E}\left[Y(a)\mid S_{2}(a),U,X,G=O\right]\mid G=E\right] \\ &=\mathbb{E}\left[\mathbb{E}\left[Y(a)\mid S_{2}(a),U,X,G=O\right]\mid G=O\right] \\ &=\mathbb{E}\left[Y(a)\mid S_{2}(a),U,X,G=O\right]\mid G=O\right] \\ &=\mathbb{E}\left[Y(a)\mid G=O\right] = \mu(a). \end{split}$$

Here the second equation uses the fact that $Y \perp S_1 \mid S_2, A, U, X, G = O$ that we prove in Lemma 6, the fourth equation uses the fact that $Y(a) \perp A \mid S_2(a), U, X, G = O$ according to Assumption 1,

the fifth equation uses the definition of the selection bridge function $q_0(S_2, S_1, A, X)$, the seventh equation uses change of measure, the eighth equation uses the fact that $A \perp (S_2(a), U, X) \mid G = E$ according to Assumption 2, and the ninth equation uses the fact that $G \perp (S_2(a), U, X)$ according to Assumption 3.

G.4 Proofs for Section 4.3

Proof for Theorem 3. If conditions in Theorem 1 hold and $h = h_0$ satisfies Equation (9), then

$$\mathbb{E} [h(S_3, S_2, A, X) \mid A = a, G = E] + \mathbb{E} [q(S_2, S_1, A, X)(Y - h(S_3, S_2, A, X)) \mid A = a, G = O]$$

$$= \mathbb{E} [h(S_3, S_2, A, X) \mid A = a, G = E]$$

$$+ \mathbb{E} [q(S_2, S_1, A, X) \mathbb{E} [Y - h(S_3, S_2, A, X) \mid S_2, S_1, A, X, G = O] \mid A = a, G = O]$$

$$= \mathbb{E} [h(S_3, S_2, A, X) \mid A = a, G = E]$$

$$= \mu(a),$$

where the second equation follows from Equation (9) and the third equation follows from Theorem 1. If conditions in Theorem 2 hold and $q = q_0$ satisfies Equation (12) or Equation (13), then

$$\mathbb{E}\left[h(S_3, S_2, A, X) \mid A = a, G = E\right] + \mathbb{E}\left[q(S_2, S_1, A, X)(Y - h(S_3, S_2, A, X)) \mid A = a, G = O\right]$$

$$= \mathbb{E}\left[q(S_2, S_1, A, X)Y \mid A = a, G = O\right]$$

$$- \mathbb{E}\left[h(S_3, S_2, A, X)\mathbb{E}\left[q(S_2, S_1, A, X) - \frac{p(S_3, S_2, X \mid A, G = E)}{p(S_3, S_2, X \mid A, G = O)} \mid S_3, S_2, A, X, G = O\right] \mid A = a, G = O\right]$$

$$= \mathbb{E}\left[q(S_2, S_1, A, X)Y \mid A = a, G = O\right]$$

$$= \mu(a),$$

where the second equation follows from Equation (12) and the third equation follows from Theorem 2. \Box

G.5 Proofs for Section 5

Proof for Theorem 4. We first prove statement (2). We define

$$\tilde{\mu}_{\text{SEL}}(a) = \frac{1}{K} \sum_{k=1}^{K} \left[\frac{1}{n_{O,k}^{(a)}} \sum_{i \in \mathcal{D}_{O,k}} \mathbb{I} \left[A_i = a \right] \tilde{q}(S_{2,i}, S_{1,i}, A_i, X_i) Y_i \right].$$

Since we assume $\tilde{q} = q_0$, as $n \to \infty$, it follows from Law of Large Number and Theorem 2 that

$$\tilde{\mu}_{\mathrm{SEL}}(a) \to \mathbb{E}\left[q_0(S_2, S_1, A, X)Y \mid A = a, G = O\right] = \mu(a).$$

Now we only need to show that $\hat{\mu}_{SEL}(a) - \tilde{\mu}_{SEL}(a) = o_p(1)$, as this would imply that $\hat{\mu}_{SEL}(a) = \mu(a) + o_p(1)$, so that $\hat{\tau}_{SEL}$ is a consistent estimator for τ . To prove this, note that

$$\hat{\mu}_{\text{SEL}}(a) - \tilde{\mu}_{\text{SEL}}(a) = \frac{1}{K} \sum_{k=1}^{K} \frac{n_{O,k}}{n_{O,k}^{(a)}} \Delta_{\text{SEL},k}.$$

where

$$\Delta_{\text{SEL},k} = \frac{1}{n_{O,k}} \sum_{i \in \mathcal{D}_{O,k}} \mathbb{I}\left[A_i = a\right] \left(\hat{q}_k(S_{2,i}, S_{1,i}, A_i, X_i) - q_0(S_{2,i}, S_{1,i}, A_i, X_i)\right) Y_i.$$

Then by Cauchy-Schwartz inequality, for any $k \in \{1, ..., K\}$, we have

$$\operatorname{Var}(\Delta_{\mathrm{SEL},k} \mid \mathcal{D}_{O,-k}) = \frac{1}{n_{O,k}} \operatorname{Var}(\mathbb{I}[A = a] (\hat{q}_k(S_2, S_1, A, X) - q_0(S_2, S_1, A, X)) Y \mid \mathcal{D}_{O,-k})$$

$$\leq \frac{1}{n_{O,k}} \mathbb{E}\left[(\mathbb{I}[A = a] (\hat{q}_k(S_2, S_1, A, X) - q_0(S_2, S_1, A, X)) Y)^2 \mid \mathcal{D}_{O,-k} \right]$$

$$\lesssim \frac{1}{n_{O,k}} \|\hat{q}_k - q_0\|_{\mathcal{L}_2(\mathbb{P})}^2 \leq \frac{\rho_{q,n}^2}{n_{O,k}}.$$

By Markov inequality, we then have that

$$|\Delta_{\mathrm{SEL},k}| = \mathbb{E}\left[|\Delta_{\mathrm{SEL},k}| \mid \mathcal{D}_{O,-k}\right] + O_p\left(\frac{\rho_{q,n}}{\sqrt{n_{O,k}}}\right).$$

Here

$$\mathbb{E}\left[|\Delta_{\mathrm{SEL},k}| \mid \mathcal{D}_{O,-k}\right] \lesssim \|\hat{q}_k - q_0\|_{\mathcal{L}_2(\mathbb{P})} = \rho_{q,n}.$$

Therefore

$$\hat{\mu}_{\text{SEL}}(a) - \tilde{\mu}_{\text{SEL}}(a) = \frac{1}{K} \sum_{k=1}^{K} \frac{n_{O,k}}{n_{O,k}^{(a)}} \Delta_{\text{SEL},k} = \frac{1}{K} \sum_{k=1}^{K} \frac{1}{\mathbb{P}(A = a \mid G = O)} O_p \left(\rho_{q,n} + \frac{\rho_{q,n}}{\sqrt{n_{O,k}}} \right)$$
$$= o_p(1).$$

Similarly, we can prove that $\hat{\mu}_{OTC}(a) = \mu(a) + o_p(1)$ so that $\hat{\tau}_{OTC}$ is a consistent estimator for τ , *i.e.*, statement (1) is true.

Finally, we can similarly prove that $\hat{\mu}_{DR}(a) - \tilde{\mu}_{DR}(a) = o_p(1)$, where

$$\tilde{\mu}_{DR}(a) = \frac{1}{K} \sum_{k=1}^{K} \left[\frac{1}{n_E^{(a)}} \sum_{i \in \mathcal{D}_E} \mathbb{I} \left[A_i = a \right] \tilde{h}(S_{3,i}, S_{2,i}, A_i, X_i) \right]$$

$$+ \frac{1}{K} \sum_{k=1}^{K} \left[\frac{1}{n_{O,k}^{(a)}} \sum_{i \in \mathcal{D}_{O,k}} \mathbb{I} \left[A_i = a \right] \tilde{q}(S_{2,i}, S_{1,i}, A_i, X_i) \left(Y_i - \tilde{h}(S_{3,i}, S_{2,i}, A_i, X_i) \right) \right].$$

By Law of Large Number, the limit of $\tilde{\mu}_{DR}(a)$ is

$$\mathbb{E}\left[\tilde{h}(S_3, S_2, A, X) \mid A = a, G = E\right] + \mathbb{E}\left[\tilde{q}(S_2, S_1, A, X)\left(Y - \tilde{h}(S_3, S_2, A, X)\right) \mid A = a, G = O\right].$$

According to Theorem 3, this is equal to $\mu(a)$ if either $\tilde{q} = q_0$ or $\tilde{h} = h_0$. Thus if either $\tilde{q} = q_0$ or $\tilde{h} = h_0$, $\hat{\mu}_{DR}(a) - \mu(a) = o_p(1)$ so that $\hat{\tau}_{DR}$ is a consistent estimator for τ . This proves statement (3).

Proof for Theorem 5. By simple algebra, we can show that

$$\hat{\mu}_{\mathrm{DR}}(a) - \tilde{\mu}_{\mathrm{DR}}(a) = \frac{1}{K} \sum_{k=1}^{K} \frac{n_{O,k}}{n_{O,k}^{(a)}} \Delta_{\mathrm{DR},k}^{O} + \frac{n_{E}}{n_{E}^{(a)}} \Delta_{\mathrm{DR},k}^{E}$$

$$= \frac{1}{K} \sum_{k=1}^{K} \frac{1}{\mathbb{P}(A = a \mid G = O) + o_{p}(1)} \Delta_{\mathrm{DR},k}^{O} + \frac{1}{\mathbb{P}(A = a \mid G = E) + o_{p}(1)} \Delta_{\mathrm{DR},k}^{E}$$

where

$$\Delta_{\mathrm{DR},k}^{O} = \frac{1}{n_{O,k}} \sum_{i \in \mathcal{D}_{O,k}} \left[\mathbb{I} \left[A_i = a \right] \hat{q}_k(S_{2,i}, S_{1,i}, A_i, X_i) \left(Y_i - \hat{h}_k(S_{3,i}, S_{2,i}, A_i, X_i) \right) - \mathbb{I} \left[A_i = a \right] q_0(S_{2,i}, S_{1,i}, A_i, X_i) \left(Y_i - h_0(S_{3,i}, S_{2,i}, A_i, X_i) \right) \right],$$

and

$$\Delta_{\mathrm{DR},k}^E = \frac{1}{n_E} \sum_{i \in \mathcal{D}_E} \mathbb{I}\left[A_i = a\right] \left(\hat{h}_k(S_{3,i}, S_{2,i}, A_i, X_i) - h_0(S_{3,i}, S_{2,i}, A_i, X_i)\right).$$

By following the proof for Theorem 4, we can show that

$$\Delta_{\mathrm{DR},k}^{O} = \mathbb{E}\left[\Delta_{\mathrm{DR},k}^{O} \mid \mathcal{D}_{O,-k}\right] + O_p\left(\frac{\max\{\rho_{q,n},\rho_{h,n}\}}{\sqrt{n_{O,k}}}\right) = \mathbb{E}\left[\Delta_{\mathrm{DR},k}^{O} \mid \mathcal{D}_{O,-k}\right] + o_p\left(n^{-1/2}\right)$$

and

$$\Delta_{\mathrm{DR},k}^{E} = \mathbb{E}\left[\Delta_{\mathrm{DR},k}^{E} \mid \mathcal{D}_{O,-k}\right] + O_{p}\left(\frac{\rho_{h,n}}{\sqrt{n_{E}}}\right) = \mathbb{E}\left[\Delta_{\mathrm{DR},k}^{E} \mid \mathcal{D}_{O,-k}\right] + o_{p}\left(n^{-1/2}\right).$$

Moreover, we have

$$\left| \frac{1}{\mathbb{P}(A=a \mid G=O)} \mathbb{E}\left[\Delta_{DR,k}^{O} \mid \mathcal{D}_{O,-k} \right] + \frac{1}{\mathbb{P}(A=a \mid G=E)} \mathbb{E}\left[\Delta_{DR,k}^{E} \mid \mathcal{D}_{O,-k} \right] \right|
= \left| \mathbb{E}\left[\hat{h}_{k}(S_{3}, S_{2}, A, X) - h_{0}(S_{3}, S_{2}, A, X) \mid A=a, G=E, \mathcal{D}_{O,-k} \right] \right|
+ \mathbb{E}\left[\hat{q}_{k}(S_{2}, S_{1}, A, X) \left(Y - \hat{h}_{k}(S_{3}, S_{2}, A, X) \right) \mid A=a, G=O, \mathcal{D}_{O,-k} \right]
- \mathbb{E}\left[q_{0}(S_{2}, S_{1}, A, X) (Y - h_{0}(S_{3}, S_{2}, A, X)) \mid A=a, G=O, \mathcal{D}_{O,-k} \right] \right|
= \left| \mathcal{R}_{k,1} + \mathcal{R}_{k,2} + \mathcal{R}_{k,3} \right|.$$
(31)

Here

$$\mathcal{R}_{k,1} = \mathbb{E}\left[q_0(S_2, S_1, A, X)\left(\hat{h}_k(S_3, S_2, A, X) - h_0(S_3, S_2, A, X)\right) \mid A = a, G = O, \mathcal{D}_{O, -k}\right]$$

$$\mathcal{R}_{k,2} = \mathbb{E}\left[\hat{q}_k(S_2, S_1, A, X)\left(h_0(S_3, S_2, A, X) - \hat{h}_k(S_3, S_2, A, X)\right) \mid A = a, G = O, \mathcal{D}_{O, -k}\right]$$

$$\mathcal{R}_{k,3} = 0.$$

Thus

Equation (31) =
$$|\mathcal{R}_{k,1} + \mathcal{R}_{k,2}|$$

= $\left| \mathbb{E} \left[(q_0 - \hat{q}_k)(\hat{h}_k - h_0) \mid A = a, G = O, \mathcal{D}_{O,-k} \right] \right|$
= $\left| \mathbb{E} \left[\frac{\mathbb{I} [A = a]}{\mathbb{P} (A = a \mid G = O)} (q_0 - \hat{q}_k)(\hat{h}_k - h_0) \mid G = O, \mathcal{D}_{O,-k} \right] \right|$

It follows that

Equation (31) =
$$\left| \mathbb{E} \left[\frac{\mathbb{I} [A = a]}{\mathbb{P} (A = a \mid G = O)} \mathbb{E} [q_0 - \hat{q}_k \mid S_3, S_2, A, X, G = O] (\hat{h}_k - h_0) \mid \mathcal{D}_{O, -k} \right] \right|$$

$$\leq \|P^*(\hat{q}_k - q_0)\|_{\mathcal{L}_2(\mathbb{P})} \|\hat{h}_k - h_0\|_{\mathcal{L}_2(\mathbb{P})},$$

and

Equation (31) =
$$\left| \mathbb{E} \left[\frac{\mathbb{I}[A=a]}{\mathbb{P}(A=a \mid G=O)} (q_0 - \hat{q}_k) \mathbb{E} \left[\hat{h}_k - h_0 \mid S_2, S_1, A, X, G=O \right] \mid \mathcal{D}_{O,-k} \right] \right|$$

 $\leq \|q_0 - \hat{q}_k\|_{\mathcal{L}_2(\mathbb{P})} \|P(\hat{h}_k - h_0)\|_{\mathcal{L}_2(\mathbb{P})}.$

This means that

Equation (31)
$$\leq \min \left\{ \|P^{\star}(\hat{q}_k - q_0)\|_{\mathcal{L}_2(\mathbb{P})} \|\hat{h}_k - h_0\|_{\mathcal{L}_2(\mathbb{P})}, \|q_0 - \hat{q}_k\|_{\mathcal{L}_2(\mathbb{P})} \|P(\hat{h}_k - h_0)\|_{\mathcal{L}_2(\mathbb{P})} \right\} = o_p \left(n^{-1/2}\right).$$

Therefore, we have $\hat{\mu}_{DR}(a) - \tilde{\mu}_{DR}(a) = o_p(n^{-1/2})$. Furthermore,

$$\begin{split} \tilde{\mu}_{\mathrm{DR}}(a) &- \mu(a) \\ &= \frac{1}{K} \sum_{k=1}^{K} \left[\frac{1}{n_{E}^{(a)}} \sum_{i \in \mathcal{D}_{E}} \mathbb{I} \left[A_{i} = a \right] \left(h_{0}(S_{3,i}, S_{2,i}, A_{i}, X_{i}) - \mu(a) \right) \right] \\ &+ \frac{1}{K} \sum_{k=1}^{K} \left[\frac{1}{n_{O,k}^{(a)}} \sum_{i \in \mathcal{D}_{O,k}} \mathbb{I} \left[A_{i} = a \right] q_{0}(S_{2,i}, S_{1,i}, A_{i}, X_{i}) (Y_{i} - h_{0}(S_{3,i}, S_{2,i}, A_{i}, X_{i})) \right] \\ &= \frac{1}{\mathbb{P}(A = a \mid G = E) n_{E}} \sum_{i \in \mathcal{D}_{E}} \mathbb{I} \left[A_{i} = a \right] \left(h_{0}(S_{3,i}, S_{2,i}, A_{i}, X_{i}) - \mu(a) \right) \\ &+ \frac{1}{\mathbb{P}(A = a \mid G = O) n_{O}} \sum_{i \in \mathcal{D}_{O}} \mathbb{I} \left[A_{i} = a \right] q_{0}(S_{2,i}, S_{1,i}, A_{i}, X_{i}) (Y_{i} - h_{0}(S_{3,i}, S_{2,i}, A_{i}, X_{i})) + o_{p} \left(n^{-1/2} \right) \end{split}$$

Combine the results above, we have

$$\hat{\tau}_{DR} - \tau = \frac{1}{n_E} \sum_{i \in \mathcal{D}_E} \left[\frac{A_i - \mathbb{P}(A_i = 1 \mid G_i = E)}{\mathbb{P}(A_i = 1 \mid G_i = E) (1 - \mathbb{P}(A_i = 1 \mid G_i = E))} (h_0(S_{3,i}, S_{2,i}, A_i, X_i) - \mu(A_i)) \right] + \frac{1}{n_O} \sum_{i \in \mathcal{D}_O} \left[\frac{A_i - \mathbb{P}(A_i = 1 \mid G_i = O)}{\mathbb{P}(A_i = 1 \mid G_i = O)} q_0(S_{2,i}, S_{1,i}, A_i, X_i) (Y_i - h_0(S_{3,i}, S_{2,i}, A_i, X_i)) \right] + o_p(n^{-1/2}).$$

Then the asserted conclusion follows from Central Limit Theorem.

Proof for Theorem 6. We only need to prove that $\hat{\sigma}^2$ is a consistent estimator for σ^2 , since then we can apply Slutsky's theorem to show that as $n \to \infty$,

$$\frac{\sqrt{n}(\hat{\tau}_{DR} - \tau)}{\hat{\sigma}} \rightsquigarrow \mathcal{N}(0, 1).$$

This in turn implies the desired asymptotic coverage conclusion.

To prove the consistency of $\hat{\sigma}^2$, we first consider the following (infeasible) estimator:

$$\tilde{\sigma}^{2} = \frac{n}{n_{E}K} \sum_{k=1}^{K} \left\{ \frac{1}{n_{E}} \sum_{i \in \mathcal{D}_{E}} \left[\frac{A_{i} - \hat{\pi}_{E}}{\hat{\pi}_{E}} (h_{0}(S_{3,i}, S_{2,i}, A_{i}, X_{i}) - \hat{\mu}_{DR}(A_{i})) \right]^{2} \right\}$$

$$+ \frac{n}{n_{O}K} \sum_{k=1}^{K} \left\{ \frac{1}{n_{O,k}^{(a)}} \sum_{i \in \mathcal{D}_{O,k}} \left[\frac{A_{i} - \hat{\pi}_{O}}{\hat{\pi}_{O}} q_{0}(S_{2,i}, S_{1,i}, A_{i}, X_{i}) (Y_{i} - h_{0}(S_{3,i}, S_{2,i}, A_{i}, X_{i})) \right]^{2} \right\}$$

$$= \frac{n}{n_{E}} \left\{ \frac{1}{n_{E}} \sum_{i \in \mathcal{D}_{E}} \left[\frac{A_{i} - \hat{\pi}_{E}}{\hat{\pi}_{E}} (h_{0}(S_{3,i}, S_{2,i}, A_{i}, X_{i}) - \hat{\mu}_{DR}(A_{i})) \right]^{2} \right\}$$

$$+ \frac{n}{n_{O}} \left\{ \frac{1}{n_{O}} \sum_{i \in \mathcal{D}_{O}} \left[\frac{A_{i} - \hat{\pi}_{O}}{\hat{\pi}_{O}} q_{0}(S_{2,i}, S_{1,i}, A_{i}, X_{i}) (Y_{i} - h_{0}(S_{3,i}, S_{2,i}, A_{i}, X_{i})) \right]^{2} \right\}.$$

Since $n/n_E \to (1+\lambda)/\lambda$, $n/n_O \to 1+\lambda$, $\hat{\pi}_E \to \mathbb{P}(A=1 \mid G=E)$, and $\hat{\pi}_O \to \mathbb{P}(A=1 \mid G=O)$, we can apply Law of Large Number and Slutsky's theorem to show that $\tilde{\sigma}^2$ is a consistent estimator for σ^2 . Therefore, as long as we can prove that $\hat{\sigma}^2 - \tilde{\sigma}^2 \to 0$ as $n \to \infty$, we have $\hat{\sigma}^2 \to \sigma^2$ as $n \to \infty$, which finishes our proof.

To prove $\hat{\sigma}^2 - \hat{\tilde{\sigma}}^2 \to 0$, we define that

$$\psi_{1,i}(h) = \frac{A_i - \hat{\pi}_E}{\hat{\pi}_E} (h(S_{3,i}, S_{2,i}, A_i, X_i) - \hat{\mu}_{DR}(A_i)),$$

$$\psi_{2,i}(h, q) = \frac{A_i - \hat{\pi}_O}{\hat{\pi}_O} q(S_{2,i}, S_{1,i}, A_i, X_i) (Y_i - h(S_{3,i}, S_{2,i}, A_i, X_i)).$$

It follows that

$$\begin{aligned} \left| \hat{\sigma}^2 - \tilde{\sigma}^2 \right| &= \frac{n}{n_E K} \sum_{k=1}^K \underbrace{\left| \frac{1}{n_E} \sum_{i \in \mathcal{D}_E} \left[\psi_{1,i}^2(\hat{h}_k) - \psi_{1,i}^2(h_0) \right] \right|}_{\Delta_{1,k}} \\ &+ \frac{n}{n_O K} \sum_{k=1}^K \underbrace{\left| \frac{1}{n_{O,k}^{(a)}} \sum_{i \in \mathcal{D}_{O,k}} \left[\psi_{2,i}^2(\hat{h}_k, \hat{q}_k) - \psi_{2,i}^2(h_0, q_0) \right] \right|}_{\Delta_{2,k}}. \end{aligned}$$

We now analyze $\Delta_{1,k}$:

$$\Delta_{1,k} \leq \left| \frac{1}{n_E} \sum_{i \in \mathcal{D}_E} \left(\psi_{1,i}(\hat{h}_k) - \psi_{1,i}(h_0) \right) \left(2\psi_{1,i}(h_0) + \psi_{1,i}(\hat{h}_k) - \psi_{1,i}(h_0) \right) \right| \\
\leq \left[\frac{1}{n_E} \sum_{i \in \mathcal{D}_E} \left(\psi_{1,i}(\hat{h}_k) - \psi_{1,i}(h_0) \right)^2 \right]^{1/2} \left\{ \left[\frac{1}{n_E} \sum_{i \in \mathcal{D}_E} \left(\psi_{1,i}(\hat{h}_k) - \psi_{1,i}(h_0) \right)^2 \right]^{1/2} + 2 \left[\frac{1}{n_E} \sum_{i \in \mathcal{D}_E} \psi_{1,i}^2(h_0) \right]^{1/2} \right\}$$

Moreover, since $\mathbb{P}(A=1 \mid G=E)$ is strictly positive according to Assumption 2, we have that for

large enough $n, \hat{\pi}_E \geq \mathbb{P}(A=1 \mid G=E)/2 > 0$ with high probability. It follows that

$$\frac{1}{n_E} \sum_{i \in \mathcal{D}_E} \left(\psi_{1,i}(\hat{h}_k) - \psi_{1,i}(h_0) \right)^2 \lesssim \frac{1}{n_E} \sum_{i \in \mathcal{D}_E} \left(\hat{h}_k(S_{3,i}, S_{2,i}, A_i, X_i) - h_0(S_{3,i}, S_{2,i}, A_i, X_i) \right)^2$$

$$= \|\hat{h}_k - h_0\|_{\mathcal{L}_2(\mathbb{P})} + o_{\mathbb{P}}(1) = O_{\mathbb{P}}(\rho_{h,n}) + o_{\mathbb{P}}(1) = o_{\mathbb{P}}(1).$$

It follows that $\Delta_{1,k} = o_{\mathbb{P}}(1)$. Similarly, we can show that $\Delta_{2,k} = o_{\mathbb{P}}(1)$. These together ensure that as $n \to \infty$,

$$\hat{\sigma}^2 - \tilde{\sigma}^2 \to 0.$$

Proof for theorem 7. We consider a semiparametric model \mathcal{M}_{sp} that places no restrictions on the data distribution except the existence of a bridge function h_0 in Assumption 6. Consider a regular parametric submodel indexed by a parameter t: $\mathcal{P}_t = \{p_t(y, s, a, x, g) : t \in \mathbb{R}\}$ where $p_0(y, s, a, x, g)$ equals the true density p(y, s, a, x, g). The associated score function is denoted as $SC(y, s, a, x, g) = \partial_t \log p_t(y, s, a, x, g)|_{t=0}$. The expectation w.r.t the distribution $p_t(y, s, a, x, g)$ is denoted by \mathbb{E}_t .

By following the proof for Theorem 11 in Kallus et al. [2021], under the condition that bridge functions h_0 , q_0 uniquely exist and the linear operator T is bijective, the tangent space corresponding to \mathcal{M}_{sp} is given by

$$S = \left\{ SC(Y, S, A, X, G) = SC(S_2, S_1, A, X, G) + SC(Y, S_3 \mid S_2, S_1, A, X, G) : \\ SC(S_2, S_1, A, X, G) \in L_2(S_2, S_1, A, X, G), SC(Y, S_3 \mid S_2, S_1, A, X, G) \in L_2(Y, S_3 \mid S_2, S_1, A, X, G), \\ \mathbb{E} \left[SC(S_2, S_1, A, X, G) \right] = 0, \mathbb{E} \left[SC(Y, S_3 \mid S_2, S_1, A, X, G) \mid S_2, S_1, A, X, G \right] = 0, \\ \mathbb{E} \left[(Y - h_0(S_3, S_2, A, X)SC(Y, S_3 \mid S_2, S_1, A, X, G)) \mid S_2, S_1, A, X, G = O \right] \in \text{Range}(T) \right\}.$$

We now analyze the path differentiability of the counterfactual mean parameter $\mu_t(a)$ under a submodel distribution with parameter value t. According to Theorem 1, we have

$$\mu_t(a) = \mathbb{E}_t \left[h_t(S_3, S_2, A, X) \mid A = a, G = E \right],$$

where $h_t(S_3, S_2, A, X)$ is the corresponding outcome bridge function defined by

$$\mathbb{E}_t [Y - h_t(S_3, S_2, A, X) \mid S_2, S_1, A, X, G = O] = 0.$$

Note that we have

$$\frac{\partial}{\partial t} \mu_{t}(a)|_{t=0} = \frac{\partial}{\partial t} \mathbb{E}_{t} \left[h_{t}(S_{3}, S_{2}, A, X) \mid A = a, G = E \right]|_{t=0}
= \mathbb{E} \left[h_{0}(S_{3}, S_{2}, A, X) SC(S_{3}, S_{2}, X \mid A, G) \mid A = a, G = E \right]
+ \frac{\partial}{\partial t} \mathbb{E} \left[h_{t}(S_{3}, S_{2}, A, X) \mid A = a, G = E \right]|_{t=0}.$$
(33)

We first analyze the term in Equation (33).

$$\mathbb{E}\left[h_{0}(S_{3}, S_{2}, A, X)SC(S_{3}, S_{2}, X \mid A, G) \mid A = a, G = E\right]$$

$$= \mathbb{E}\left[\left(h_{0}(S_{3}, S_{2}, A, X) - \mu(a)\right)SC(S_{3}, S_{2}, X \mid A, G) \mid A = a, G = E\right]$$

$$= \mathbb{E}\left[\left(h_{0}(S_{3}, S_{2}, A, X) - \mu(a)\right)SC(S_{3}, S_{2}, A, X, G) \mid A = a, G = E\right]$$

$$= \mathbb{E}\left[\left(h_{0}(S_{3}, S_{2}, A, X) - \mu(a)\right)SC(Y, S_{3}, S_{2}, S_{1}, A, X, G) \mid A = a, G = E\right]$$

$$= \mathbb{E}\left[\frac{\mathbb{I}\left[A = a, G = E\right]}{\mathbb{P}\left(A = a, G = E\right)}\left(h_{0}(S_{3}, S_{2}, A, X) - \mu(a)\right)SC(Y, S_{3}, S_{2}, S_{1}, A, X, G)\right]$$

where the second equation holds because

$$\mathbb{E}\left[(h_0(S_3, S_2, A, X) - \mu(a)) SC(A, G) \mid A = a, G = E \right]$$

$$= \mathbb{E}\left[(h_0(S_3, S_2, A, X) - \mu(a)) \mid A = a, G = E \right] SC(A = a, G = E) = 0,$$

and the third equation holds because

$$\mathbb{E}\left[(h_0(S_3, S_2, A, X) - \mu(a))\operatorname{SC}(Y, S_1 \mid S_3, S_2, A, X, G) \mid A = a, G = E\right]$$

$$= \mathbb{E}\left[(h_0(S_3, S_2, A, X) - \mu(a))\mathbb{E}\left[\operatorname{SC}(Y, S_1 \mid S_3, S_2, A, X, G) \mid S_3, S_2, A, X, G\right] \mid A = a, G = E\right] = 0.$$

Next we analyze the term in Equation (34).

$$\begin{split} &\frac{\partial}{\partial t} \mathbb{E} \left[h_t(S_3, S_2, A, X) \mid A = a, G = E \right] |_{t=0} \\ &= \frac{\partial}{\partial t} \mathbb{E} \left[\frac{p(S_3, S_2, X \mid A, G = E)}{p(S_3, S_2, X \mid A, G = O)} h_t(S_3, S_2, A, X) \mid A = a, G = O \right] |_{t=0} \\ &= \frac{\partial}{\partial t} \mathbb{E} \left[q_0(S_2, S_1, A, X) h_t(S_3, S_2, A, X) \mid A = a, G = O \right] |_{t=0} \\ &= \mathbb{E} \left[q_0(S_2, S_1, A, X) \frac{\partial}{\partial t} \mathbb{E} \left[h_t(S_3, S_2, A, X) \mid S_2, S_1, A, X, G = O \right] |_{t=0} \mid A = a, G = O \right] , \end{split}$$

where the second equation holds because of Equation (12).

Furthermore, by taking the derivative of the left hand side w.r.t t at t = 0, we have

$$\frac{\partial}{\partial t} \mathbb{E} \left[h_t(S_3, S_2, A, X) \mid S_2, S_1, A, X, G = O \right] |_{t=0}$$

$$= \mathbb{E} \left[(Y - h_0(S_3, S_2, A, X)) \operatorname{SC}(Y, S_3 \mid S_2, S_1, A, X, G) \mid S_2, S_1, A, X, G = O \right] = 0.$$
(36)

It follows that

$$\frac{\partial}{\partial t} \mathbb{E} \left[h_t(S_3, S_2, A, X) \mid A = a, G = E \right] |_{t=0}$$

$$= \mathbb{E} \left[q_0(S_2, S_1, A, X) (Y - h_0(S_3, S_2, A, X)) SC(Y, S_3 \mid S_2, S_1, A, X, G) \mid A = a, G = O \right]$$

$$= \mathbb{E} \left[q_0(S_2, S_1, A, X) (Y - h_0(S_3, S_2, A, X)) SC(Y, S_3, S_2, S_1, A, X, G) \mid A = a, G = O \right]$$

$$= \mathbb{E} \left[\frac{\mathbb{I} \left[A = a, G = O \right]}{\mathbb{P} \left(A = a, G = O \right)} q_0(S_2, S_1, A, X) (Y - h_0(S_3, S_2, A, X)) SC(Y, S_3, S_2, S_1, A, X, G) \right],$$

where the second equation holds because

$$\mathbb{E}\left[q_{0}(S_{2}, S_{1}, A, X)(Y - h_{0}(S_{3}, S_{2}, A, X))SC(S_{2}, S_{1}, A, X, G) \mid A = a, G = O\right]$$

$$= \mathbb{E}\left[q_{0}(S_{2}, S_{1}, A, X)\mathbb{E}\left[Y - h_{0}(S_{3}, S_{2}, A, X) \mid S_{2}, S_{1}, A, X, G = O\right] \times SC(S_{2}, S_{1}, A, X, G = O) \mid A = a, G = O\right] = 0.$$

Combining Equations (35) and (37), we have

$$\frac{\partial}{\partial t} \mu_t(a)|_{t=0} = \mathbb{E}\left[\psi_a(Y, S_3, S_2, S_1, A, X, G)SC(Y, S_3, S_2, S_1, A, X, G)\right],$$

where

$$\begin{split} \psi_a(Y,S_3,S_2,S_1,A,X,G) &= \frac{\mathbb{I}\left[A=a,G=E\right]}{\mathbb{P}\left(A=a,G=E\right)} (h_0(S_3,S_2,A,X) - \mu(a)) \\ &+ \frac{\mathbb{I}\left[A=a,G=O\right]}{\mathbb{P}\left(A=a,G=O\right)} q_0(S_2,S_1,A,X) (Y-h_0(S_3,S_2,A,X)). \end{split}$$

Therefore,

$$\frac{\partial}{\partial t} \tau_t|_{t=0} = \frac{\partial}{\partial t} \mu_t(1)|_{t=0} - \frac{\partial}{\partial t} \mu_t(0)|_{t=0}$$
$$= \mathbb{E} \left[\psi(Y, S_3, S_2, S_1, A, X, G) \text{SC}(Y, S_3, S_2, S_1, A, X, G) \right],$$

where

$$\begin{split} &\psi(Y, S_3, S_2, S_1, A, X, G) \\ &= \psi_1(Y, S_3, S_2, S_1, A, X, G) - \psi_0(Y, S_3, S_2, S_1, A, X, G) - \tau \\ &= \frac{\mathbb{I}\left[G = E\right]}{\mathbb{P}\left(G = E\right)} \frac{A - \mathbb{P}\left(A = 1 \mid G = E\right)}{\mathbb{P}\left(A = 1 \mid G = E\right)} (h_0(S_3, S_2, A, X) - \mu(A)) \\ &+ \frac{\mathbb{I}\left[G = O\right]}{\mathbb{P}\left(G = O\right)} \frac{A - \mathbb{P}\left(A = 1 \mid G = O\right)}{\mathbb{P}\left(A = 1 \mid G = O\right)} q_0(S_2, S_1, A, X) (Y - h_0(S_3, S_2, A, X)) - \tau. \end{split}$$

We can easily decompose $\psi(Y, S_3, S_2, S_1, A, X, G)$ into two terms:

$$\psi(Y, S_3, S_2, S_1, A, X, G) = \mathbb{E} \left[\psi(Y, S_3, S_2, S_1, A, X, G) \mid S_2, S_1, A, X, G \right] - \tau$$

$$+ \psi(Y, S_3, S_2, S_1, A, X, G) - \mathbb{E} \left[\psi(Y, S_3, S_2, S_1, A, X, G) \mid S_2, S_1, A, X, G \right],$$

where

$$\begin{split} & \mathbb{E}\left[\psi(Y, S_3, S_2, S_1, A, X, G) \mid S_2, S_1, A, X, G\right] - \tau \in L_2(S_2, S_1, A, X, G) \\ & \mathbb{E}\left[\mathbb{E}\left[\psi(Y, S_3, S_2, S_1, A, X, G) \mid S_2, S_1, A, X, G\right] - \tau\right] = 0 \\ & \psi(Y, S_3, S_2, S_1, A, X, G) - \mathbb{E}\left[\psi(Y, S_3, S_2, S_1, A, X, G) \mid S_2, S_1, A, X, G\right] \in L_2(Y, S_3 \mid S_2, S_1, A, X, G) \\ & \mathbb{E}\left[\psi(Y, S_3, S_2, S_1, A, X, G) - \mathbb{E}\left[\psi(Y, S_3, S_2, S_1, A, X, G) \mid S_2, S_1, A, X, G\right] \mid S_2, S_1, A, X, G\right] = 0. \end{split}$$

Moreover, since T is surjective, its range space Range(T) is the whole $L_2(S_2, S_1, A, X)$ space so we automatically have

$$\mathbb{E}\big[(\psi(Y, S_3, S_2, S_1, A, X, G) - \mathbb{E}\left[\psi(Y, S_3, S_2, S_1, A, X, G) \mid S_2, S_1, A, X, G\right]) \times (Y - h_0(S_3, S_2, A, X)) \mid S_2, S_1, A, X, G = O\big] \in \text{Range}(T).$$

This means that $\psi(Y, S_3, S_2, S_1, A, X, G)$ belongs to the tangent space \mathcal{S} . Thus $\psi(Y, S_3, S_2, S_1, A, X, G)$ is the efficient influence function for τ , and its variance, which is equal to σ^2 in Theorem 7, is the semiparametric efficiency lower bound for τ relative to the tangent space \mathcal{S} in Equation (32).

G.6 Proofs for Section 6

Proof for Theorem 8. Before proving the theorem, we note that by Bayes rule, we can easily verify that

$$\frac{\mathbb{P}\left(G=E\mid A=a\right)\mathbb{P}\left(G=O\mid X\right)}{\mathbb{P}\left(G=O\mid A=a\right)\mathbb{P}\left(G=E\mid X\right)}\frac{\mathbb{I}\left[A=a\right]}{\mathbb{P}\left(A=a\mid X,G=E\right)}=\frac{\mathbb{I}\left[A=a\right]}{\mathbb{P}\left(A=a\mid X,G=O\right)}\frac{p(X\mid A=a,G=O)}{p(X\mid A=a,G=E)}.$$

Assume that condition 1 holds so we have $h = h_0$ satisfying Equation (9). In this case, for any function q, we have

$$\mathbb{E}\left[\frac{\mathbb{P}(G = E \mid A = a) \,\mathbb{P}(G = O \mid X)}{\mathbb{P}(G = O \mid A = a) \,\mathbb{P}(G = E \mid X)} \frac{\mathbb{I}[A = a]}{\mathbb{P}(A = a \mid X, G = E)} q\left(S_{2}, S_{1}, A, X\right) \left(Y - h\left(S_{3}, S_{2}, A, X\right)\right) \mid G = O\right]$$

$$= \mathbb{E}\left[\frac{\mathbb{P}(G = E \mid A = a) \,\mathbb{P}(G = O \mid X)}{\mathbb{P}(G = O \mid A = a) \,\mathbb{P}(G = E \mid X)} \frac{\mathbb{I}[A = a]}{\mathbb{P}(A = a \mid X, G = E)} q\left(S_{2}, S_{1}, A, X\right) \right]$$

$$\times \mathbb{E}\left[Y - h_{0}\left(S_{3}, S_{2}, A, X\right) \mid S_{2}, S_{1}, A, X, G = O\right] \mid G = O\right] = 0, (38)$$

where the last equation uses the conditional moment equation in Equation (9). Moreover, for function $h = h_0$,

$$\mathbb{E}\left[\frac{\mathbb{P}(G=E)\,\mathbb{P}(G=O\mid X)}{\mathbb{P}(G=O)\,\mathbb{P}(G=E\mid X)}\frac{\mathbb{I}[A=a]}{\mathbb{P}(A=a\mid X,G=E)}\left(h(S_3,S_2,A,X)-\bar{h}_E(A,X)\right)\mid G=E\right]$$

$$=\mathbb{E}\left[\frac{\mathbb{P}(G=E)\,\mathbb{P}(G=O\mid X)}{\mathbb{P}(G=O)\,\mathbb{P}(G=E\mid X)}\mathbb{E}\left[h(S_3,S_2,a,X)-\bar{h}_E(a,X)\mid A=a,X,G=E\right]\mid G=E\right]=0.$$
(39)

Finally, we only need to prove that

$$\mu(a) = \mathbb{E}\left[\mathbb{E}\left[h_0(S_3, S_2, A, X) \mid A = a, X = x, G = E\right] \mid G = O\right] \tag{40}$$

According to lemma 1, we already know that any function $h_0(S_3, S_2, A, X)$ that satisfies Equation (9) must be a valid bridge function in the sense of Equation (7). Thus we only need to prove Equation (40) for $h_0(S_3, S_2, A, X)$ that satisfies Equation (7). By following the proof in Theorem 1, we can show that

$$\mathbb{E}[h_0(S_3, S_2, A, X) \mid A = a, X, G = E] = \mathbb{E}[\mathbb{E}[Y(a) \mid S_2(a), U, X, G = O] \mid A = a, X, G = E].$$

Therefore,

$$\mathbb{E} \left[\mathbb{E} \left[h_0(S_3, S_2, A, X) \mid A = a, X, G = E \right] \mid G = O \right] \\ = \mathbb{E} \left[\mathbb{E} \left[\mathbb{E} \left[Y(a) \mid S_2(a), U, X, G = O \right] \mid X, G = E \right] \mid G = O \right] \\ = \mathbb{E} \left[\mathbb{E} \left[\mathbb{E} \left[Y(a) \mid S_2(a), U, X, G = O \right] \mid X, G = O \right] \mid G = O \right] \\ = \mathbb{E} \left[\mathbb{E} \left[Y(a) \mid X, G = O \right] \mid G = O \right] = \mathbb{E} \left[Y(a) \mid G = O \right].$$

Here the first equation follows from the fact that $A \perp (S(a), U) \mid X, G = E$ in Assumption 10, the second equation follows from Equation (30) in Lemma 7, and the third equation follows from the fact that $G \perp (S(a), U) \mid X$ in Assumption 9.

Combining Equations (38) to (40) proves the conclusion.

Assume that condition 2 holds so we have $q = q_0$ satisfying Equation (12) or Equation (13). We first prove that

$$\mu(a) = \mathbb{E}\left[\frac{\mathbb{I}[A=a]}{\mathbb{P}(A=a \mid X, G=O)} \frac{p(X \mid A=a, G=O)}{p(X \mid A=a, G=E)} q_0(S_2, S_1, A, X) Y \mid G=O\right]$$

$$= \mathbb{E}\left[\frac{\mathbb{P}(G=E \mid A=a) \mathbb{P}(G=O \mid X)}{\mathbb{P}(G=O \mid A=a) \mathbb{P}(G=E \mid X)} \frac{\mathbb{I}[A=a]}{\mathbb{P}(A=a \mid X, G=E)} q(S_2, S_1, A, X) Y \mid G=O\right].$$
(41)

To prove this, note that according to Lemma 2, any function $q_0(S_2, S_1, A, X)$ that satisfies Equation (12) or Equation (13) is a valid selection bridge function in the sense of Equation (11). Thus we only need to prove Equation (41) for any q_0 that satisfies Equation (11). We further note that

the right hand side of Equation (41) is equal to the following:

$$\begin{split} &\mathbb{E}\left[\mathbb{E}\left[\frac{p(X\mid A=a,G=O)}{p(X\mid A=a,G=E)}q_{0}\left(S_{2},S_{1},A,X\right)Y\mid A=a,X,G=O\right]\mid G=O\right] \\ &=\mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[\frac{p(X\mid A=a,G=O)}{p(X\mid A=a,G=E)}q_{0}\left(S_{2},S_{1},A,X\right)\mid S_{2},A=a,U,X,G=O\right]\right.\\ &\times \mathbb{E}\left[Y\mid S_{2},A=a,U,X,G=O\right]\mid A=a,X,G=O\right]\mid G=O\right] \\ &=\mathbb{E}\left[\mathbb{E}\left[\frac{p(S_{2},U\mid A,X,G=E)}{p(S_{2},U\mid A,X,G=O)}\mathbb{E}\left[Y(a)\mid S_{2}(a),U,X,G=O\right]\mid A=a,X,G=O\right]\mid G=O\right] \\ &=\mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[Y(a)\mid S_{2}(a),U,X,G=O\right]\mid A=a,X,G=E\right]\mid G=O\right] \\ &=\mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[Y(a)\mid S_{2}(a),U,X,G=O\right]\mid X,G=E\right]\mid G=O\right] \\ &=\mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[Y(a)\mid S_{2}(a),U,X,G=O\right]\mid X,G=E\right]\mid G=O\right] \\ &=\mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[Y(a)\mid S_{2}(a),U,X,G=O\right]\mid X,G=O\right]\mid X,G=O\right] \\ \\ &=\mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[Y(a)\mid S_{2}(a)$$

Here the first equation uses $Y \perp S_1 \mid S_2, A, U, X, G = O$ which we prove in Lemma 6, the second equation uses the fact that q_0 satisfies Equation (11) and $Y(a) \perp A \mid S_2(a), U, X, G = O$ according to Assumption 1, the fourth equation uses that $S_2(a) \perp A \mid X, G = E$ according to Assumption 10, the fifth equation uses the fact that $S_2(a) \perp G \mid X$ according to Assumption 9.

Next, we can follow the proof above to show that for any h,

$$\mathbb{E}\left[\frac{\mathbb{I}[A=a]}{\mathbb{P}(A=a\mid X, G=O)} \frac{p(X\mid A=a, G=O)}{p(X\mid A=a, G=E)} q_0(S_2, S_1, A, X) h(S_3, S_2, A, X) \mid G=O\right]$$

$$=\mathbb{E}\left[h(S_3(a), S_2(a), a, X) \mid G=O\right]$$
(42)

And by change of measure, we can also verify that

$$\mathbb{E}\left[\frac{\mathbb{P}(G=E)\,\mathbb{P}(G=O\mid X)}{\mathbb{P}(G=O)\,\mathbb{P}(G=E\mid X)}\frac{\mathbb{I}[A=a]}{\mathbb{P}(A=a\mid X,G=E)}h(S_3,S_2,A,X)\mid G=E\right]$$

$$=\mathbb{E}\left[\mathbb{E}\left[h(S_3,S_2,A,X)\mid A=a,X,G=E\right]\mid G=O\right] = \mathbb{E}\left[h(S_3(a),S_2(a),a,X)\mid G=O\right], \quad (43)$$

and

$$\mathbb{E}\left[\frac{\mathbb{P}\left(G=E\right)\mathbb{P}\left(G=O\mid X\right)}{\mathbb{P}\left(G=O\right)\mathbb{P}\left(G=E\mid X\right)}\frac{\mathbb{I}\left[A=a\right]}{\mathbb{P}\left(A=a\mid X,G=E\right)}\bar{h}_{E}(A,X)\mid G=E\right]\\ =&\mathbb{E}\left[\mathbb{E}\left[\bar{h}_{E}(A,X)\mid A=a,X,G=E\right]\mid G=O\right]=\mathbb{E}\left[\bar{h}_{E}(A,X)\mid G=O\right]$$

These show that

$$0 = \mathbb{E} \left[h_{E}(a, X) \mid G = O \right]$$

$$+ \mathbb{E} \left[\frac{\mathbb{P} (G = E) \mathbb{P} (G = O \mid X)}{\mathbb{P} (G = O) \mathbb{P} (G = E \mid X)} \frac{\mathbb{I} [A = a]}{\mathbb{P} (A = a \mid X, G = E)} \left(h(S_{3}, S_{2}, A, X) - \bar{h}_{E}(A, X) \right) \mid G = E \right]$$

$$- \mathbb{E} \left[\frac{\mathbb{I} [A = a]}{\mathbb{P} (A = a \mid X, G = O)} \frac{p(X \mid A = a, G = O)}{p(X \mid A = a, G = E)} q_{0} (S_{2}, S_{1}, A, X) h(S_{3}, S_{2}, A, X) \mid G = O \right].$$
 (44)

Combining Equations (41) and (44) leads to the conclusion.

G.7 Proofs for Appendix

Proof for Proposition 1. First note that

$$\begin{split} \frac{p(S_2, U, X \mid A = a, G = E)}{p(S_2, U, X \mid A = a, G = O)} &= \frac{p(U, X \mid A = a, G = E)}{p(U, X \mid A = a, G = O)} \\ &= \frac{\mathbb{P}(A = a \mid U, X, G = E)}{\mathbb{P}(A = a \mid U, X, G = O)} \frac{\mathbb{P}(A = a \mid G = O)}{\mathbb{P}(A = a \mid G = O)} \\ &= \frac{\mathbb{P}(A = a \mid G = O)}{\mathbb{P}(A = a \mid U, X, G = O)}, \end{split}$$

where the first equation follows from Lemma 7, the second equation follows from Bayes rule, and the third equation follows from the fact that $\mathbb{P}(A=a\mid U,X,G=E)=\mathbb{P}(A=a\mid G=E)=\frac{1}{2}$. Therefore, we have

$$\frac{p(S_2, U, X \mid A = a, G = E)}{p(S_2, U, X \mid A = a, G = O)} = \frac{\mathbb{E}\left[\left[1 + \exp\left((-1)^a \left(\kappa_1^\top U + \kappa_2^\top X\right)\right)\right]^{-1}\right]}{\left[1 + \exp\left((-1)^a \left(\kappa_1^\top U + \kappa_2^\top X\right)\right)\right]^{-1}}.$$
(45)

Second, $(S_1, S_2) \mid A, U, X, G = O$ follows a joint Gaussian distribution whose conditional expectation is

$$\begin{bmatrix} \tau_1 A + \beta_1 X + \gamma_1 U \\ (\tau_2 + \alpha_2 \tau_1) A + (\beta_2 + \alpha_2 \beta_1) X + (\gamma_2 + \alpha_2 \gamma_1) U \end{bmatrix}$$

and conditional covariance matrix is

$$\begin{bmatrix} \sigma_1^2 I_1 & \sigma_1^2 \alpha_2^\top \\ \sigma_1^2 \alpha_2 & \sigma_1^2 \alpha_2 \alpha_2^\top + \sigma_2^2 I_2 \end{bmatrix}.$$

It follows that $S_1 \mid S_2, A, U, X, G = O$ also has a Gaussian distribution function with conditional expectation

$$\lambda_1 S_2 + \lambda_2 A + \lambda_3 X + \lambda_4 U$$

and conditional variance

$$\Sigma_{1|2} = \sigma_1^2 I_1 - \sigma_1^4 \alpha_2^{\top} \left(\sigma_1^2 \alpha_2 \alpha_2^{\top} + \sigma_2^2 I_2 \right)^{-1} \alpha_2.$$

where

$$\begin{split} \lambda_1 &= \sigma_1^2 \alpha_2^\top \left(\sigma_1^2 \alpha_2 \alpha_2^\top + \sigma_2^2 I_2 \right)^{-1}, \\ \lambda_2 &= \left(I_1 - \sigma_1^2 \alpha_2^\top \left(\sigma_1^2 \alpha_2 \alpha_2^\top + \sigma_2^2 I_2 \right)^{-1} \alpha_2 \right) \tau_1 - \sigma_1^2 \alpha_2^\top \left(\sigma_1^2 \alpha_2 \alpha_2^\top + \sigma_2^2 I_2 \right)^{-1} \tau_2 \\ \lambda_3 &= \left(I_1 - \sigma_1^2 \alpha_2^\top \left(\sigma_1^2 \alpha_2 \alpha_2^\top + \sigma_2^2 I_2 \right)^{-1} \alpha_2 \right) \beta_1 - \sigma_1^2 \alpha_2^\top \left(\sigma_1^2 \alpha_2 \alpha_2^\top + \sigma_2^2 I_2 \right)^{-1} \beta_2 \\ \lambda_4 &= \left(I_1 - \sigma_1^2 \alpha_2^\top \left(\sigma_1^2 \alpha_2 \alpha_2^\top + \sigma_2^2 I_2 \right)^{-1} \alpha_2 \right) \gamma_1 - \sigma_1^2 \alpha_2^\top \left(\sigma_1^2 \alpha_2 \alpha_2^\top + \sigma_2^2 I_2 \right)^{-1} \gamma_2. \end{split}$$

Third, for a = 1, we posit a selection bridge function of the following form:

$$q_0(S_2, S_1, 1, X) = c_1 \exp\left(\tilde{\theta}_2^{\top} S_2 + \tilde{\theta}_1^{\top} S_1 + \tilde{\theta}_0^{\top} X\right) + c_0.$$

It follows that

$$\mathbb{E}\left[q_{0}(S_{2}, S_{1}, 1, X) \mid S_{2}, A = 1, U, X, G = O\right]$$

$$=c_{1} \exp\left(\tilde{\theta}_{2}^{\top} S_{2} + \tilde{\theta}_{0}^{\top} X\right) \mathbb{E}\left[\exp\left(\tilde{\theta}_{1}^{\top} S_{1}\right) \mid S_{2}, A = 1, U, X, G = O\right] + c_{0}$$

$$=c_{1} \exp\left(\tilde{\theta}_{2}^{\top} S_{2} + \tilde{\theta}_{0}^{\top} X\right) \exp\left(\tilde{\theta}_{1}^{\top} (\lambda_{1} S_{2} + \lambda_{2} A + \lambda_{3} X + \lambda_{4} U) + \frac{1}{2} \tilde{\theta}_{1}^{\top} \Sigma_{1|2} \tilde{\theta}_{1}\right) + c_{0}$$

$$=c_{1} \exp\left(\frac{1}{2} \tilde{\theta}_{1}^{\top} \Sigma_{1|2} \tilde{\theta}_{1}\right) \exp\left(\left(\tilde{\theta}_{1}^{\top} \lambda_{1} + \tilde{\theta}_{2}^{\top}\right) S_{2} + \tilde{\theta}_{1}^{\top} \lambda_{2} A + \left(\tilde{\theta}_{1}^{\top} \lambda_{3} + \tilde{\theta}_{0}^{\top}\right) X + \tilde{\theta}_{1}^{\top} \lambda_{4} U\right) + c_{0}$$

Thus we only need the above to match Equation (45) for a=1. This is possible once λ_4 has full column rank: then there exists $\tilde{\theta}_1$ such that $\tilde{\theta}_1^{\top}\lambda_4 = \kappa_2^{\top}$. Then we can choose $\tilde{\theta}_1, \tilde{\theta}_0, c_1, c_0$ accordingly. Analogously, we can also show the existence of a selection bridge function $q_0(S_2, S_1, 0, X)$ of the same form for a=0.

Proof for Corollary 2. We first prove the conclusion for Equation (10) in Theorem 1. Following the proof for Theorem 1, we have

$$\mathbb{E} [h_0(S_3, S_2, A, X) \mid A = a, G = E]$$

$$= \mathbb{E} [\mathbb{E} [Y(a) \mid S_2(a), U, X, G = O] \mid G = E]$$

$$= \mathbb{E} [\mathbb{E} [Y(a) \mid S_2(a), U, X, G = E] \mid G = E] = \mathbb{E} [Y(a) \mid G = E],$$

where the second follows from the assumption that $Y(a) \perp G \mid S(a), U, X$.

Next, we prove the conclusion for Equation (14) in Theorem 2. Following the proof for Theorem 2, we have

$$\mathbb{E} [q_0(S_2, S_1, A, X)Y \mid A = a, G = O]$$

$$= \mathbb{E} [\mathbb{E} [Y(a) \mid S_2(a), U, X, G = O] \mid A = a, G = E]$$

$$= \mathbb{E} [\mathbb{E} [Y(a) \mid S_2(a), U, X, G = E] \mid G = E]$$

$$= \mathbb{E} [Y(a) \mid G = E] = \mu(a),$$

where the second equation follows from the assumption that $Y(a) \perp G \mid S(a), U, X$.

Finally, according to the proof of Theorem 3, if conditions in Theorem 1 hold and $h = h_0$ satisfies Equation (9), then

$$\mathbb{E}\left[h(S_3, S_2, A, X) \mid A = a, G = E\right] + \mathbb{E}\left[q(S_2, S_1, A, X)(Y - h(S_3, S_2, A, X)) \mid A = a, G = O\right]$$

$$= \mathbb{E}\left[h(S_3, S_2, A, X) \mid A = a, G = E\right].$$

If conditions in Theorem 2 hold and $q = q_0$ satisfies Equation (12) or Equation (13), then

$$\mathbb{E}\left[h(S_3, S_2, A, X) \mid A = a, G = E\right] + \mathbb{E}\left[q(S_2, S_1, A, X)(Y - h(S_3, S_2, A, X)) \mid A = a, G = O\right]$$

$$= \mathbb{E}\left[q(S_2, S_1, A, X)Y \mid A = a, G = O\right].$$

Then the conclusion follows from our proof above.

Proof for Corollary 3. The proof for Corollary 3 straitforwardly follows from the proof for Theorem 8 and Corollary 1 by replacing all Y with r(Y).

Proof for Lemma 3. We denote the map in Equation (23) as $\Phi(\eta)$. Then we need to prove that

$$\dot{\Phi}_j(\eta^*)[\eta_j - \eta_j^*] := \frac{\partial}{\partial t} \Phi(\eta_1^*, \dots, \eta_j^* + t(\eta_j - \eta_j^*), \dots, \eta_7^*)|_{t=0} = 0, \text{ for any } \eta_j \text{ and } j \in \{1, \dots, 7\}.$$

First, we note that

$$\begin{split} &\dot{\Phi}_{1}(\eta^{*})[\eta_{1}-\eta_{1}^{*}] \\ &= \sum_{a\in\{0,1\}} (-1)^{1-a} \bigg\{ \mathbb{E} \bigg[\frac{\mathbb{P}(G=E)\,\mathbb{P}(G=O\mid X)}{\mathbb{P}(G=O\mid X)} \frac{\mathbb{I}[A=a]}{\mathbb{P}(A=a\mid X,G=E)} (h-h_{0})(S_{3},S_{2},a,X) \mid G=E \bigg] \\ &- \mathbb{E} \bigg[\frac{\mathbb{P}(G=E\mid A=a)\,\mathbb{P}(G=O\mid X)}{\mathbb{P}(G=O\mid A=a)\,\mathbb{P}(G=E\mid X)} \frac{\mathbb{I}[A=a]}{\mathbb{P}(A=a\mid X,G=E)} q_{0}\left(S_{2},S_{1},a,X\right) \left(h-h_{0}\right)(S_{3},S_{2},a,X) \mid G=O \bigg] \bigg\} \\ &= \sum_{a\in\{0,1\}} (-1)^{1-a} \left\{ \mathbb{E} \left[(h-h_{0})(S_{3}(a),S_{2}(a),a,X) \mid G=O \right] - \mathbb{E} \left[(h-h_{0})(S_{3}(a),S_{2}(a),a,X) \mid G=O \right] \right\} = 0, \end{split}$$

where the second equation follows from Equations (42) and (43) in the proof for Theorem 8. Second, we have that

$$\begin{split} \dot{\Phi}_{2}(\eta^{*})[\eta_{2}-\eta_{2}^{*}] &= \sum_{a\in\{0,1\}} (-1)^{1-a} \bigg\{ \mathbb{E}\left[\bar{h}_{E}(a,X) - \bar{h}_{0,E}(a,X) \mid G = O\right] \\ &- \mathbb{E}\left[\frac{\mathbb{P}\left(G=E\right)\mathbb{P}\left(G=O\mid X\right)}{\mathbb{P}\left(G=O\mid X\right)} \frac{\mathbb{I}\left[A=a\right]}{\mathbb{P}\left(A=a\mid X,G=E\right)} \left(\bar{h}_{E}(a,X) - \bar{h}_{E,0}(a,X)\right) \mid G = E\right] \right\} \\ &= \sum_{a\in\{0,1\}} (-1)^{1-a} \left\{ \mathbb{E}\left[\bar{h}_{E}(a,X) - \bar{h}_{0,E}(a,X) \mid G = O\right] - \mathbb{E}\left[\bar{h}_{E}(a,X) - \bar{h}_{0,E}(a,X) \mid G = O\right] \right\} = 0, \end{split}$$

where the equation follows from the proof for Theorem 8.

Third, we have

$$\dot{\Phi}_{3}(\eta^{*})[\eta_{3} - \eta_{3}^{*}] = \sum_{a \in \{0,1\}} (-1)^{1-a} \mathbb{E} \left[\frac{\mathbb{P}(G = E \mid A = a) \mathbb{P}(G = O \mid X)}{\mathbb{P}(G = O \mid A = a) \mathbb{P}(G = E \mid X)} \frac{\mathbb{I}[A = a]}{\mathbb{P}(A = a \mid X, G = E)} \right] \\
\times (q - q_{0})(S_{2}, S_{1}, a, X) (Y - h_{0}(S_{3}, S_{2}, A, X)) \mid G = O \right] \\
= \sum_{a \in \{0,1\}} (-1)^{1-a} \mathbb{E} \left[\frac{\mathbb{P}(G = E \mid A = a) \mathbb{P}(G = O \mid X)}{\mathbb{P}(G = O \mid A = a) \mathbb{P}(G = E \mid X)} \frac{\mathbb{I}[A = a]}{\mathbb{P}(A = a \mid X, G = E)} \right] \\
\times (q - q_{0})(S_{2}, S_{1}, a, X) \mathbb{E}[Y - h_{0}(S_{3}, S_{2}, A, X) \mid S_{2}, S_{1}, A = a, X, G = O) \mid G = O \right] \\
= 0.$$

Fourth, we have

$$\begin{split} \dot{\Phi}_{4}(\eta^{*})[\eta_{4} - \eta_{4}^{*}] \\ &= \sum_{a \in \{0,1\}} (-1)^{1-a} \bigg\{ \mathbb{E} \bigg[\frac{\mathbb{P} (G = E) \, \mathbb{P} (G = O \mid X)}{\mathbb{P} (G = O) \, \mathbb{P} (G = E \mid X)} \\ &\times \frac{\mathbb{I} [A = a]}{\mathbb{P}^{2} (A = a \mid X, G = E)} (\eta_{4}^{*} - \eta_{4}) \mathbb{E} \left[h_{0}(S_{3}, S_{2}, a, X) - \bar{h}_{E,0}(a, X) \mid A = a, X, G = O \right] \mid G = O \bigg] \\ &+ \mathbb{E} \bigg[\frac{\mathbb{P} (G = E \mid A = a) \, \mathbb{P} (G = O \mid X)}{\mathbb{P} (G = O \mid X) \, \mathbb{P}^{2} (A = a \mid X, G = E)} (\eta_{4}^{*} - \eta_{4}) \\ &\times q_{0} (S_{2}, S_{1}, a, X) \, \mathbb{E} [Y - h_{0} (S_{3}, S_{2}, A, X) \mid S_{2}, S_{1}, A = a, X] \mid G = O \bigg] = 0. \end{split}$$

Following this proof for $\dot{\Phi}_4(\eta^*)[\eta_4 - \eta_4^*] = 0$, we can similarly show that $\dot{\Phi}_j(\eta^*)[\eta_j - \eta_j^*] = 0$ for j = 5, 6, 7.

Proof for Theorem 10. Given the asserted conditions, according to Theorem 3.1 in Chernozhukov et al. [2019], we have

$$\hat{\tau} - \tau = \frac{1}{n_O} \sum_{i \in \mathcal{D}_O} (\phi_1(Y_i, S_i, 1, X_i; \eta^*) - \phi_1(Y_i, S_i, 0, X_i; \eta^*) - \tau) + (\phi_3(Y_i, S_i, 1, X_i; \eta^*) - \phi_3(Y_i, S_i, 0, X_i; \eta^*)) + \frac{1}{n_E} \sum_{i \in \mathcal{D}_E} (\phi_2(Y_i, S_i, 1, X_i; \eta^*) - \phi_2(Y_i, S_i, 0, X_i; \eta^*)).$$

Then the asserted conclusion follows from central limit theorem.

Proof for corollary 4. We can first follow the proof for Theorem 1 to show that for any $h_0(S_3, S_2, A, X)$ that satisfies Equation (7),

$$\mathbb{E} [h_0(S_3, S_2, A, X) \mid S_1, A = a, X, G = E]$$

$$= \mathbb{E} [\mathbb{E} [Y(a) \mid S_2(a), U, X, G = O] \mid S_1, A = a, X, G = E].$$

The rest of the proof is analogous to Corollary 1.

Proof for corollary 5. We can first follow the proof for Theorem 1 to show that for any $h_0(S_3, S_2, A, X)$ that satisfies Equation (7),

$$\mathbb{E} [h_0(S_3, S_2, A, X) \mid S_2, S_1, A = a, X, G = E]$$

$$= \mathbb{E} [\mathbb{E} [Y(a) \mid S_2(a), U, X, G = O] \mid S_2, S_1, A = a, X, G = E].$$

The rest of the proof is analogous to Corollary 1.

Proof for Corollary 6. First, note that under Assumptions 15 and 16, we can follow the proofs for Lemmas 5 and 6 to show that $S_3 \perp G \mid S_2, A = a, U, X$, and $(Y, S_3) \perp S_1 \mid S_2, A, U_{\diamond}, X, G = O$.

Second, following the proof for Lemma 1, we can show that for any function h_0 that satisfies Equation (9), it must also satisfy

$$\mathbb{E}[Y \mid S_2, A, U_{\diamond}, X, G = O] = \mathbb{E}[h_0(S_3, S_2, A, X) \mid S_2, A, U_{\diamond}, X, G = O]. \tag{46}$$

Finally, we can follow the proof for Corollary 1 to show that for any function h_0 that satisfies Equation (46), Equation (20) in Corollary 1 holds. This concludes the proof for Corollary 6.

Proof of Proposition 3. We already have $Z_2 \perp G \mid Z_1$. Thus, we only need to verify $G \perp Z_1$. Note that

$$\begin{split} p(z_1 \mid G = 1) = & p(z_1, A = 1 \mid G = 1) + p(z_1, A = 0 \mid G = 1) \\ = & \frac{\mathbb{P}\left(G = 1 \mid A = 1, Z_1 = z_1\right) \mathbb{P}\left(A = 1\right) p(z_1)}{\mathbb{P}\left(G = 1\right)} \\ & + \frac{\mathbb{P}\left(G = 1 \mid A = 0, Z_1 = z_1\right) \mathbb{P}\left(A = 0\right) p(z_1)}{\mathbb{P}\left(G = 1\right)} \\ = & p(z_1) \frac{C}{\mathbb{P}\left(G = 1\right)} \propto p(z_1), \end{split}$$

which proves the desired result.

Proof of Theorem 11. From the definition of external validity bridge function, we have

$$\frac{p(U \mid S_2, X, A = a, G = O)}{p(U \mid S_2, X, A = a, G = E)} = \mathbb{E}[\tilde{q}(S_1, S_2, X, A) \mid S_2, X, U, A = a, G = E] \cdot \frac{p(S_2, X \mid A = a, G = E)}{p(S_2, X \mid A = a, G = O)}.$$

Then

$$\begin{split} &p(U \mid S_2, X, A = a, G = O) \\ &= \mathbb{E}[\tilde{q}(S_1, S_2, X, A) \mid S_2, X, U, A = a, G = E] \cdot \frac{p(S_2, X \mid A = a, G = E)}{p(S_2, X \mid A = a, G = O)} \cdot p(U \mid S_2, X, A = a, G = E) \\ &= \mathbb{E}[\tilde{q}(S_1, S_2, X, A) \mid S_2, X, U, A = a, G = E] \cdot \frac{p(S_2, X \mid G = E)}{p(S_2, X \mid A = a, G = O)} \cdot p(U \mid S_2, X, G = E), \end{split}$$

where for the second equality we use $(S_2, X, U) \perp A \mid G = E$. Then

$$p(U \mid S_{2}, X, G = O)$$

$$= \sum_{a} p(U \mid A = a, S_{2}, X, G = O) \mathbb{P} (A = a \mid S_{2}, X, G = O)$$

$$= \sum_{a} \mathbb{P} (A = a \mid S_{2}, X, G = O) \mathbb{E} [\tilde{q}(S_{1}, S_{2}, X, A) \mid S_{2}, X, U, A = a, G = E]$$

$$\cdot \frac{p(S_{2}, X \mid G = E)}{p(S_{2}, X \mid A = a, G = O)} \cdot p(U \mid S_{2}, X, G = E)$$

$$= \sum_{a} \mathbb{E} [\tilde{q}(S_{2}, X, A, Z) \mid S_{2}, X, U, A = a, G = E] \mathbb{P} (A = a \mid G = O) \cdot \frac{p(S_{2}, X \mid G = E)}{p(S_{2}, X \mid G = O)} \cdot p(U \mid S_{2}, X, G = E)$$

$$= \mathbb{E} \left[\frac{\mathbb{P} (A \mid G = O)}{\mathbb{P} (A \mid G = E)} \tilde{q}(S_{1}, S_{2}, X, A) \mid X, U, G = E \right] \cdot \frac{p(S_{2}, X \mid G = E)}{p(S_{2}, X \mid G = O)} \cdot p(U \mid S_{2}, X, G = E), \quad (48)$$

where to get (47) we use that

$$\frac{\mathbb{P}(A = a \mid S_2, X, G = O)}{p(S_2, X \mid A = a, G = O)} = \frac{\mathbb{P}(A = a \mid G = O)}{p(S_2, X \mid G = O)}$$

and for the last equality we use again that $A \perp (S_2, U, X) \mid G = E$ so that

$$\begin{split} &\sum_{a} \mathbb{E}[\tilde{q}(S_{1}, S_{2}, X, A) \mid S_{2}, X, U, A = a, G = E] \mathbb{P}\left(A = a \mid G = O\right) \\ &= \sum_{a} \mathbb{E}[\tilde{q}(S_{1}, S_{2}, X, A) \mid S_{2}, X, U, A = a, G = E] \frac{\mathbb{P}\left(A = a \mid G = O\right)}{\mathbb{P}\left(A = a \mid G = E\right)} \mathbb{P}\left(A = a \mid G = E\right) \\ &= \sum_{a} \mathbb{E}[\tilde{q}(S_{1}, S_{2}, X, A) \mid S_{2}, X, U, A = a, G = E] \frac{\mathbb{P}\left(A = a \mid G = O\right)}{\mathbb{P}\left(A = a \mid G = E\right)} \mathbb{P}\left(A = a \mid S_{2}, X, U, G = E\right) \\ &= \mathbb{E}\left[\frac{\mathbb{P}\left(A \mid G = O\right)}{\mathbb{P}\left(A \mid G = E\right)} \tilde{q}(S_{1}, S_{2}, X, A) \mid S_{2}, X, U, G = E\right]. \end{split}$$

From above, we have that

$$\begin{split} & \mathbb{E}[Y(a) \mid S_2, X, G = O] = \mathbb{E}[\mathbb{E}[Y(a) \mid S_2, U, X, G = O] \mid S_2, X, G = O] \\ & = \mathbb{E}[\mathbb{E}[Y(a) \mid S_2, U, X, G = E] \mid S_2, X, G = O] \\ & = \mathbb{E}[\mathbb{E}[h(S_3, S_2, X, A) \mid A = a, S_2, U, X, G = E] \mid S_2, X, G = O] \\ & = \mathbb{E}\left[\mathbb{E}[h(S_3, S_2, X, A) \mid A = a, S_2, U, X, G = E] \frac{p(U \mid S_2, X, G = O)}{p(U \mid S_2, X, G = E)} \mid S_2, X, G = E\right] \end{split}$$

Where for the second equality we use that $G \perp Y(a) \mid S_2, U, X$ Then from (48), we further have

$$\begin{split} \mathbb{E}[Y(a) \mid S_2, X, G = O] &= \\ \mathbb{E}\left[\mathbb{E}[h(S_3, S_2, X, A) \mid U, S_2, X, G = E, A = a] \mathbb{E}\left[\frac{p(A \mid G = O)}{p(A \mid G = E)}\tilde{q}(S_1, S_2, X, A) \mid S_2, X, U, G = E\right] \mid S_2, X, G = E\right] \\ &\cdot \frac{p(S_2, X \mid G = E)}{p(S_2, X \mid G = O)}. \end{split}$$

From here, and that for any function $f(S_1, S_2, X, A)$,

$$\mathbb{E}\left[\mathbb{E}[h(S_{3},S_{2},X,A)\mid U,S_{2},X,G=E,A=a]\mathbb{E}\left[f(S_{1},S_{2},X,A)\mid S_{2},X,U,G=E\right]\mid S_{2},X,G=E\right]$$

$$=\mathbb{E}\left[\mathbb{E}[h(S_{3},S_{2},X,A)\mid U,S_{2},X,G=E,A=a]\right]$$

$$\cdot\mathbb{E}\left[\sum_{a'}f(S_{1},S_{2},X,a')\mathbb{P}\left(A=a'\mid G=E\right)\mid S_{2},X,U,G=E\right]\mid S_{2},X,G=E\right]$$

$$=\mathbb{E}\left[\mathbb{E}[h(S_{3},S_{2},X,A)\mid U,S_{2},X,G=E,A=a]\right]$$

$$\cdot\mathbb{E}\left[\sum_{a'}f(S_{1},S_{2},X,a')\mathbb{P}\left(A=a'\mid G=E\right)\mid S_{2},X,U,A=a,G=E\right]\mid S_{2},X,G=E,A=a\right]$$

$$=\mathbb{E}\left[h(S_{3},S_{2},X,A)\sum_{a'}f(S_{1},S_{2},X,a')\mathbb{P}\left(A=a'\mid G=E\right)\mid S_{2},X,U,A=a,G=E\right],$$

where for the second equality we use that $S_1 \perp A \mid S_2, X, U, G = E$ and that $U \perp A \mid S_2, X, G = E$.

Finally, we have that

$$\begin{split} & \mathbb{E}[Y(a) \mid S_{2}, X, G = O] \\ & = \mathbb{E}\left[h(S_{3}, S_{2}, X, A) \sum_{a'} \frac{\mathbb{P}(A = a' \mid G = O)}{\mathbb{P}(A = a' \mid G = E)} \tilde{q}(S_{1}, S_{2}, X, a') \mathbb{P}(A = a' \mid G = E) \mid S_{2}, X, A = a, G = E\right] \\ & \cdot \frac{p(S_{2}, X \mid G = E)}{p(S_{2}, X \mid G = O)} \\ & = \mathbb{E}\left[h(S_{3}, S_{2}, X, A) \sum_{a'} \mathbb{P}(A = a' \mid G = O) \tilde{q}(S_{1}, S_{2}, X, a') \mid S_{2}, X, A = a, G = E\right] \cdot \frac{p(S_{2}, X \mid G = E)}{p(S_{2}, X \mid G = O)}, \end{split}$$

which proves the desired result. It then follows that

$$\mu(a) = \mathbb{E}[\mathbb{E}[Y(a) \mid S_2, X, G = O] \mid G = O] = \mathbb{E}\left[m(S_2, a, X) \frac{p(S_2, X \mid G = E)}{p(S_2, X \mid G = O)} \mid G = O\right]$$
$$= \mathbb{E}[m(S_2, a, X) \mid G = E].$$

Therefore, we have

$$\tau = \mu(1) - \mu(0) = \mathbb{E}[m(S_2, 1, X) - m(S_2, 0, X) \mid G = E].$$

Proof for Theorem 9. We first define the following stochastic processes:

$$\mathcal{V}(a) = \left\{ p_{S_3(a)}(s_3 \mid S_2(a), S_1(a), A, X, G = O) : s_3 \in \mathcal{S}_3 \right\},$$

$$\tilde{\mathcal{V}}(a) = \left\{ p_{S_3(a)}(s_3 \mid S_2(a), S_1(a), A = a, X, G = O) : s_3 \in \mathcal{S}_3 \right\},$$

$$\mathcal{W}(a) = \left\{ p(u \mid S_2(a), S_1(a), A, X, G = O) : u \in \mathcal{U} \right\},$$

$$\tilde{\mathcal{W}}(a) = \left\{ p(u \mid S_2(a), S_1(a), A = a, X, G = O) : u \in \mathcal{U} \right\}.$$

To prove the desired conclusion, we only need to prove that

$$\mathbb{E}[r(Y(a)) \mid X, G = O] = \mathbb{E}[\mathbb{E}[r(Y) \mid \mathcal{V}, S_2, A, X, G = O] \mid A = a, X, G = E]. \tag{49}$$

Then the conclusion follows from the iterated law of conditional expectation. We will prove Equation (49) above by showing that

$$\mathbb{E}\left[\mathbb{E}\left[r(Y) \mid \mathcal{V}, S_2, A, X, G = O\right] \mid A = a, X, G = E\right]$$

$$= \mathbb{E}\left[\mathbb{E}\left[r(Y(a)) \mid \mathcal{V}(a), S_2(a), X, G = O\right] \mid A = a, X, G = E\right]. \tag{50}$$

Then Equation (49) follows from the fact that

$$\mathbb{E}\left[\mathbb{E}\left[r(Y(a)) \mid \mathcal{V}(a), S_2(a), X, G = O\right] \mid A = a, X, G = E\right]$$

$$= \mathbb{E}\left[\mathbb{E}\left[r(Y(a)) \mid \tilde{\mathcal{V}}(a), S_2(a), X, G = O\right] \mid A = a, X, G = E\right]$$

$$= \mathbb{E}\left[\mathbb{E}\left[r(Y(a)) \mid \tilde{\mathcal{V}}(a), S_2(a), X, G = O\right] \mid X, G = E\right]$$

$$= \mathbb{E}\left[\mathbb{E}\left[r(Y(a)) \mid \tilde{\mathcal{V}}(a), S_2(a), X, G = O\right] \mid X, G = O\right]$$

$$= \mathbb{E}\left[r(Y(a)) \mid X, G = O\right],$$

where the second equality follows from the fact that $(S_2(a), S_1(a)) \perp A \mid X, G = E$ and that $\tilde{\mathcal{V}}(a)$ is determined by $(S_2(a), S_1(a), X)$, the third equality follows from the fact that $(S_2(a), S_1(a)) \perp G = O \mid X$, and the last equality follows from the iterated law of conditional expectation.

Now we focus on proving Equation (50). For brevity, we omit X in all derivations so all conditional expectations below should be understood as conditioning on X implicitly. We prove Equation (50) in two steps. **Step I:** we first derive the relation between $\tilde{W}(a)$ and $\tilde{V}(a)$ and the relation between W(a) and V(a), under the completeness condition in Assumption 5 condition 1. By the law of total probability, we have

$$\begin{split} p_{S_3(a)}(s_3 \mid S_2(a) &= s_2, S_1(a) = s_1, A = a, G = O) \\ &= \int p_{S_3(a)}(s_3 \mid S_2(a) = s_2, S_1(a) = s_1, A = a, U = u, G = O) p(u \mid S_2(a) = s_2, S_1(a) = s_1, A = a, G = O) \, \mathrm{d}u \\ &= \int p_{S_3(a)}(s_3 \mid S_2(a) = s_2, U = u, G = O) p(u \mid S_2(a) = s_2, S_1(a) = s_1, A = a, G = O) \, \mathrm{d}u \\ &= \Phi_{s_2} \left[p_U(\cdot \mid S_2(a) = s_2, S_1(a) = s_1, A = a, G = O) \right] (s_3), \end{split}$$

where the second equality follows from the fact that $S_3(a) \perp (S_1(a), A) \mid X, U, G = O$, and Φ_{s_2} is a mapping defined as follows: for any function $g : \mathcal{U} \to \mathbb{R}$,

$$\phi_{s_2}[g(u)](s_3) = \int p_{S_3(a)}(s_3 \mid S_2(a) = s_2, U = u, G = O)g(u) du.$$

Now we show that this mapping is injective. To see this, consider any two functions $g_1: \mathcal{U} \to \mathbb{R}$ and $g_2: \mathcal{U} \to \mathbb{R}$ such that $\Phi_{s_2}[g_1](s_3) = \Phi_{s_2}[g_2](s_3)$ for all s_3 such that $p(s_3 \mid S_2 = s_2, A = a, G = O) > 0$. Note that we have

$$\phi_{s_2}[g(u)](s_3) = \int p_{S_3(a)}(s_3 \mid S_2(a) = s_2, U = u, G = O)g(u) du$$

$$= \int p_{S_3(a)}(s_3 \mid S_2(a) = s_2, A = a, U = u, G = O)g(u) du$$

$$= \int p_{S_3}(s_3 \mid S_2 = s_2, A = a, U = u, G = O)g(u) du$$

$$= \int p(u \mid S_3 = s_3, S_2 = s_2, A = a, G = O) \frac{p(s_3 \mid S_2 = s_2, A = a, G = O)}{p(u \mid S_2 = s_2, A = a, G = O)} g(u) du.$$

According to the completeness condition in Assumption 5 condition 1, $\phi_{s_2}[g(u)](s_3) = 0$ for all s_3 such that $p(s_3 \mid S_2 = s_2, A = a, G = O) > 0$ if and only if g(u) = 0 for all u such that $p(u \mid S_2 = s_2, A = a, G = O) > 0$. This in turn implies that $\Phi_{s_2}[g_1](s_3) - \Phi_{s_2}[g_2](s_3) = \Phi_{s_2}[g_1 - g_2](s_3) = 0$ for all s_3 such that $p(s_3 \mid S_2 = s_2, A = a, G = O) > 0$ if and only if $g_1(u) = g_2(u)$ for all u such that $p(u \mid S_2 = s_2, A = a, G = O) > 0$. Therefore, ϕ_{s_2} is an injective mapping. It follows that there exists another mapping Ψ_{s_2} such that

$$p(u \mid s_2(a) = s_2, S_1(a) = s_1, A = a, G = O) = \Psi_{s_2}[p_{S_3(a)}(s_3 \mid S_2(a) = s_2, S_1(a) = s_1, A = a, G = O)](u).$$

Therefore, we have $\tilde{\mathcal{W}}(a) = \Psi_{S_2(a)}[\tilde{\mathcal{V}}(a)].$

By the same token, we also have

$$p_{S_3(a)}(s_3 \mid S_2(a) = s_2, S_1(a) = s_1, A, G = O) = \Phi_{s_2}[p_U(\cdot \mid S_2(a) = s_2, S_1(a) = s_1, A, G = O)](s_3).$$

We thus also have $W(a) = \Psi_{S_2(a)}[V(a)].$

Step II: We next prove Equation (50). Note that

$$\mathbb{E}\left[\mathbb{E}\left[r(Y) \mid \mathcal{V}, S_{2}, A, G = O\right] \mid A = a, G = E\right] = \mathbb{E}\left[\mathbb{E}\left[r(Y(a)) \mid \mathcal{V}(a), S_{2}(a), A = a, G = O\right] \mid A = a, G = E\right].$$

By the iterated law of conditional expectation,

$$\begin{split} &\mathbb{E}\left[r(Y(a)) \mid \mathcal{V}(a), S_{2}(a), A = a, G = O\right] \\ =&\mathbb{E}\left[\mathbb{E}\left[r(Y(a)) \mid \mathcal{V}(a), S_{2}(a), S_{1}(a), A = a, G = O\right] \mid \mathcal{V}(a), S_{2}(a), A = a, G = O\right] \\ =&\mathbb{E}\left[\mathbb{E}\left[r(Y(a)) \mid \mathcal{V}(a), S_{2}(a), S_{1}(a), A, G = O\right] \mid \mathcal{V}(a), S_{2}(a), A = a, G = O\right] \\ =&\mathbb{E}\left[\mathbb{E}\left[r(Y(a)) \mid S_{2}(a), S_{1}(a), A, G = O\right] \mid \mathcal{V}(a), S_{2}(a), A = a, G = O\right] \\ =&\mathbb{E}\left[\mathbb{E}\left[r(Y(a)) \mid S_{2}(a), S_{1}(a), A = a, G = O\right] \mid \mathcal{V}(a), S_{2}(a), A = a, G = O\right] \\ =&\mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[r(Y(a)) \mid S_{2}(a), S_{1}(a), A = a, U, G = O\right] \mid S_{2}(a), S_{1}(a), A = a, G = O\right] \mid \mathcal{V}(a), S_{2}(a), A = a, G = O\right] \\ =&\mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[r(Y(a)) \mid S_{2}(a), U, G = O\right] \mid S_{2}(a), S_{1}(a), A = a, G = O\right] \mid \mathcal{V}(a), S_{2}(a), A = a, G = O\right], \end{split}$$

where the thid equality holds because V(a) is fully determined by $S_2(a), S_1(a), A$, and the last equality holds because $Y(a) \perp S_1(a) \mid S_2(a), U, X, G = O$.

Here

$$\mathbb{E}\left[\mathbb{E}\left[r(Y(a)) \mid S_{2}(a), U, G = O\right] \mid S_{2}(a), S_{1}(a), A = a, G = O\right]$$

$$= \int \mathbb{E}\left[r(Y(a)) \mid S_{2}(a), U = u, G = O\right] p(u \mid S_{2}(a), S_{1}(a), A = a, G = O) du$$

$$= \int \mathbb{E}\left[r(Y(a)) \mid S_{2}(a), U = u, G = O\right] [\tilde{\mathcal{W}}(a)](u) du$$

$$= \int \mathbb{E}\left[r(Y(a)) \mid S_{2}(a), U = u, G = O\right] [\Psi_{S_{2}(a)}[\tilde{\mathcal{V}}(a)]](u) du$$

It then follows that

$$\begin{split} &\mathbb{E}\left[\mathbb{E}\left[r(Y(a))\mid \mathcal{V}(a), S_2(a), A=a, G=O\right]\mid A=a, G=E\right] \\ &=\mathbb{E}\left[\mathbb{E}\left[\int \mathbb{E}\left[r(Y(a))\mid S_2(a), U=u, G=O\right] \left[\Psi_{S_2(a)}[\tilde{\mathcal{V}}(a)]\right](u) \, \mathrm{d}u \mid \mathcal{V}(a), S_2(a), A=a, G=O\right] \mid A=a, G=E\right] \\ &=\mathbb{E}\left[\mathbb{E}\left[\int \mathbb{E}\left[r(Y(a))\mid S_2(a), U=u, G=O\right] \left[\Psi_{S_2(a)}[\mathcal{V}(a)]\right](u) \, \mathrm{d}u \mid \mathcal{V}(a), S_2(a), A, G=O\right] \mid A=a, G=E\right] \\ &=\mathbb{E}\left[\mathbb{E}\left[\int \mathbb{E}\left[r(Y(a))\mid S_2(a), U=u, G=O\right] \left[\Psi_{S_2(a)}[\mathcal{V}(a)]\right](u) \, \mathrm{d}u \mid \mathcal{V}(a), S_2(a), G=O\right] \mid A=a, G=E\right]. \end{split}$$

where the last equality follows from the fact that conditionally on $S_2(a)$, the inner term within $\mathbb{E}\left[\cdot\mid \mathcal{V}(a), S_2(a), A = a, G = O\right]$ in the second equality above only depends on V(a). Moreover, we have

 $\int \mathbb{E}[r(Y(a)) \mid S_{2}(a), U = u, G = O] [\Psi_{S_{2}(a)}[\mathcal{V}(a)]](u) du$ $= \int \mathbb{E}[r(Y(a)) \mid S_{2}(a), S_{1}(a), A, U = u, G = O] [\mathcal{W}(a)](u) du$ $= \int \mathbb{E}[r(Y(a)) \mid S_{2}(a), S_{1}(a), A, U = u, G = O] p(u \mid S_{2}(a), S_{1}(a), A, G = O) du$ $= \int \mathbb{E}[r(Y(a)) \mid \mathcal{V}(a), S_{2}(a), S_{1}(a), A, U = u, G = O] p(u \mid \mathcal{V}(a), S_{2}(a), S_{1}(a), A, G = O) du$

$$= \int \mathbb{E}\left[r(Y(a)) \mid \mathcal{V}(a), S_2(a), S_1(a), A, U = u, G = O\right] p(u \mid \mathcal{V}(a), S_2(a), S_1(a), A, G = O) du$$

 $= \mathbb{E}\left[r(Y(a)) \mid \mathcal{V}(a), S_2(a), S_1(a), G = O\right],$

where the first equality uses the fact that $Y(a) \perp (S_1(a), A) \mid S_2(a), U, X, G = O$, the third equality uses the fact that $\mathcal{V}(a)$ is fully determined by $S_2(a), S_1(a), A$, and the last equality uses the iterated law of conditional expectation.

This means that

$$\mathbb{E}\left[\int \mathbb{E}\left[r(Y(a)) \mid S_{2}(a), U = u, G = O\right] \left[\Psi_{S_{2}(a)}[\mathcal{V}(a)]\right](u) \, \mathrm{d}u \mid \mathcal{V}(a), S_{2}(a), G = O\right]$$

$$= \mathbb{E}\left[\mathbb{E}\left[r(Y(a)) \mid \mathcal{V}(a), S_{2}(a), S_{1}(a), G = O\right] \mid \mathcal{V}(a), S_{2}(a), G = O\right]$$

$$= \mathbb{E}\left[r(Y(a)) \mid \mathcal{V}(a), S_{2}(a), G = O\right].$$

It follows that

$$\mathbb{E}\left[\mathbb{E}\left[r(Y)\mid\mathcal{V},S_{2},A,G=O\right]\mid A=a,G=E\right]=\mathbb{E}\left[\mathbb{E}\left[r(Y(a))\mid\mathcal{V}(a),S_{2}(a),G=O\right]\mid A=a,G=E\right].$$

This finishes proving Equation (50) (with X being implicitly conditioned on everywhere).